ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\geq\frac{9}{4}\cdot\sqrt{\frac{6R-3r}{5R-r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$w_a^2=\frac{4bc}{(b+c)^2}.s(s-a)=\frac{bc((b+c)^2-a^2)}{(b+c)^2}=bc-\frac{a^2bc}{(b+c)^2}\overset{A-G}{\geq}bc-\frac{a^2}{4}$$

$$\Rightarrow\frac{w_a^2}{ab}\geq\frac{c}{a}-\frac{a}{4b}\text{ and analogs}\Rightarrow\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\geq\sum_{cyc}\frac{c}{a}-\frac{1}{4}.\sum_{cyc}\frac{a}{b}=\frac{3}{4}.\sum_{cyc}\frac{a}{b}$$

$$\geq\frac{3}{4}.\frac{9\sum_{cyc}a^2}{\left(\sum_{cyc}a\right)^2}\overset{?}{\geq}\frac{9}{4}.\sqrt{\frac{6R-3r}{5R-r}}\Leftrightarrow\frac{9.4(s^2-4Rr-r)^2}{16s^4}\overset{?}{\geq}\frac{6R-3r}{5R-r}$$

$$\Leftrightarrow(15R-3r)(s^2-4Rr-r)^2\overset{?}{\geq}(8R-4r)s^4$$

$$\Leftrightarrow(7R+r)s^4-r(120R^2+6Rr-6r^2)s^2+r^2(240R^3+72R^2r-9Rr^2-3r^3)\overset{(1)}{\geq}0$$

and $\because(7R+r)s^4\overset{\text{Gerretsen}}{\geq}(7R+r)(s^2-16Rr+5r^2)^2\therefore$ in order to prove (1),
it suffices to prove : LHS of (1) $\geq(7R+r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(26R^2-11Rr-r^2)s^2\overset{(2)}{\geq}r(388R^3-234R^2r+6Rr^2+7r^3)$$

Now, $(26R^2-11Rr-r^2)s^2\overset{\text{Gerretsen}}{\geq}(26R^2-11Rr-r^2)(16Rr-5r^2)\overset{?}{\geq}$
$r(388R^3-234R^2r+6Rr^2+7r^3)\Leftrightarrow 28t^3-72t^2+33t-2\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$

$$\Leftrightarrow(t-2)(20t^2+8t(t-2)+1)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$\therefore\frac{w_a^2}{ab}+\frac{w_b^2}{bc}+\frac{w_c^2}{ca}\geq\frac{9}{4}.\sqrt{\frac{6R-3r}{5R-r}}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{w_a{}^2}{ab}=\sum_{cyc}\frac{4cs(s-a)}{a(b+c)^2}\overset{AM-GM}{\geq}\sum_{cyc}\frac{27cs(s-a)}{\left(a+\frac{b+c}{2}+\frac{b+c}{2}\right)^3}=\frac{27}{8s^2}\sum_{cyc}c(s-a)=$$

$$=\frac{27(s^2-r^2-4Rr)}{8s^2}=\frac{27}{8}\left(1-\frac{r(4R+r)}{s^2}\right)\overset{Gerretsen}{\geq}$$

$$\geq \frac{27}{8}\left(1-\frac{r(4R+r)}{16Rr-5r^2}\right)=\frac{81(2R-r)}{4(16R-5r)}\geq$$

$$\overset{?}{\geq}\frac{9}{4}\sqrt{\frac{6R-3r}{5R-r}}\Leftrightarrow 27(2R-r)(5R-r)\overset{?}{\geq}(16R-5r)^2 \Leftrightarrow (R-2r)(14R-r)\overset{?}{\geq}0,$$

which is true by Euler's inequality $R\geq 2r$. Equality holds iff ΔABC is equilateral.