ROMANIAN MATHEMATICAL MAGAZINE

In *any* Δ ABC, the following relationship holds :

$$R + r - \sqrt{R^2 - 4r^2} \le h_a \le w_a \le R + r + \sqrt{R(R - 2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}} = \frac{2R(c^2 - s^2)}{c}\ \left(c = \cos\frac{B-C}{2}\ \textit{and}\ s = \sin\frac{A}{2}\right)$$

$$\le R + r + \sqrt{R(R-2r)} = R + 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) + R.\sqrt{1 - 4sc + 4s^2}$$

$$\Leftrightarrow 2c - \frac{2s^2}{c} \le 1 + 2sc - 2s^2 + \sqrt{1 - 4sc + 4s^2}$$

$$\Leftrightarrow 1 + 2sc - 2c + \frac{2s^2}{c} - 2s^2 + \sqrt{1 - 4sc + 4s^2} \overset{(1)}{\ge} 0$$

$$\text{Now,} \frac{2s^2}{c} - 2s^2 = \frac{2s^2\left(1 - \cos\frac{B-C}{2}\right)}{\cos\frac{B-C}{2}} \ge 0\ \textit{and}\ 1 - 4sc + 4s^2 \overset{0<c\le 1}{\ge} 1 - 4s + 4s^2$$

$$= (1-2s)^2 \Rightarrow \sqrt{1 - 4sc + 4s^2} \ge |1 - 2s|\ \textit{and so, in order to prove}\ (1),$$

$$\text{it suffices to prove :}\ \boxed{1 + 2sc - 2c + |1 - 2s| \overset{(2)}{\ge} 0}$$

$\boxed{\textit{Case 1}}$ $1 - 2s \ge 0$ *and* then : LHS of (2) $= 1 + 2sc - 2c + 1 - 2s$

$$= 1 - c - c(1-2s) + 1 - 2s = (1-2s)(1-c) + (1-c) = 2(1-c)(1-s) \ge 0$$

$$\because c = \cos\frac{B-C}{2} \le 1\ \textit{and}\ s = \sin\frac{A}{2} < 1 \Rightarrow (2)\ \text{is true}$$

$\boxed{\textit{Case 2}}$ $1 - 2s < 0$ *and* then : LHS of (2) $= 1 + 2sc - 2c + 2s - 1$

$$= 1 - c + c(2s-1) + (2s-1) \overset{c\le 1}{\ge} (2s-1)(1+c) > 0 \because 1 - 2s < 0$$

$$\Rightarrow (2)\ \text{is true (strict inequality)}$$

$$\therefore \text{combining both cases, (2) is true}\ \forall\ \Delta\ ABC \therefore w_a \le R + r + \sqrt{R(R-2r)}$$

$$\text{Now,} \sqrt{R^2 - 4r^2} = \sqrt{(R-2r)(R+2r)} = \sqrt{R(1 - 4sc + 4s^2)\big(R + 4Rs(c-s)\big)}$$

$$= R.\sqrt{(1 - 4sc + 4s^2)\big(1 + 4s(c-s)\big)} \ge R.\sqrt{1 - 4sc + 4s^2}$$

$$\left(\because c = \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2} = s\right) \therefore h_a + \sqrt{R^2 - 4r^2} \ge$$

$$2R(c^2 - s^2) + R.\sqrt{1 - 4sc + 4s^2} \overset{?}{\ge} R + r = R + 2Rs(c - s)$$

$$\Leftrightarrow \boxed{\sqrt{1-4sc+4s^2} \underbrace{\overset{?}{\geq}}_{③} 1+2sc-2c^2}$$ and its's trivially true when

$1+2sc-2c^2 < 0$ and so we now focus on the scenario when :

$1+2sc-2c^2 \geq 0$ and then : ③ $\Leftrightarrow 1-4sc+4s^2 \geq (1+2sc-2c^2)^2$

$\Leftrightarrow -c^4+2c^3s-c^2s^2+c^2-2cs+s^2 \geq 0 \Leftrightarrow -c^2(c-s)^2+(c-s)^2 \geq 0$

$\Leftrightarrow (c-s)^2(1-c^2) \geq 0 \to$ true $\because 1 \geq \cos^2\frac{B-C}{2} \Rightarrow$ ③ is true

$\therefore h_a \geq R+r-\sqrt{R^2-4r^2}$ and so, $R+r-\sqrt{R^2-4r^2} \leq h_a \leq w_a \leq R+r+\sqrt{R(R-2r)}\ \forall \Delta ABC$, " = " iff $b = c$ (QED)