ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{288}{w_a w_b w_c} \le \frac{36}{r_a r_b r_c} + \frac{243}{h_a h_b h_c} + \frac{8\sqrt{3}}{abc}$$

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$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}, w_b = \frac{2\sqrt{ac}}{a+c}\sqrt{r_a r_c}, w_c = \frac{2\sqrt{ab}}{a+b}\sqrt{r_a r_b}$$

$$\frac{w_a w_b w_c}{r_a r_b r_c} = \frac{8abc}{(a+b)(b+c)(c+a)} = \frac{8.4Rrs}{2s(s^2+r^2+4Rr)-4Rrs} =$$

$$= \frac{16Rr}{s^2+2Rr+r^2} \overset{Gerretsen}{\ge} \frac{16Rr}{4R^2+4Rr+3r^2+2Rr+r^2} = \frac{16Rr}{4R^2+6Rr+4r^2} =$$

$$= \frac{16\left(\frac{R}{r}\right)}{4\left(\frac{R}{r}\right)^2+6\left(\frac{R}{r}\right)+4} \overset{\frac{R}{r}=x\ge 2}{=} \frac{16x}{4x^2+6x+4} \quad (1)$$

$$\frac{r_a r_b r_c}{h_a h_b h_c} = s^2 r.\frac{4Rrs}{8r^3s^3} = \frac{R}{2r} \overset{\frac{R}{r}=x\ge 2}{=} \frac{x}{2} \quad (2)$$

$$\frac{8\sqrt{3}r_a r_b r_c}{abc} \ge \frac{8\sqrt{3}s^2 r}{4Rrs} \overset{Mitrinovic}{\ge} \frac{8\sqrt{3}}{4}\frac{3\sqrt{3}r}{R} = \frac{18r}{R} \overset{\frac{R}{r}=x\ge 2}{=} \frac{18}{x} \quad (3)$$

We need to show:

$$\frac{288}{w_a w_b w_c} \le \frac{36}{r_a r_b r_c} + \frac{243}{h_a h_b h_c} + \frac{8\sqrt{3}}{abc}$$

$$288\frac{r_a r_b r_c}{w_a w_b w_c} \le 36 + \frac{r_a r_b r_c}{h_a h_b h_c}243 + \frac{8\sqrt{3}r_a r_b r_c}{abc}$$

$$Using\ (1),(2)\&(3), \frac{4x^2+6x+4}{16x}288 \le 36 + 243.\frac{x}{2} + \frac{18}{x}$$

$$\frac{2}{x}(4x^2+6x+4) \le 4 + \frac{27x}{2} + \frac{2}{x}$$

$$16x^2+24x+16 \le 8x+27x^2+4 or, 11x^2-16x-12 \ge 0$$

$$(x-2)(11x+6) \ge 0\ true\ as\ x \ge 2\ Euler$$

$$Equality\ for\ a=b=c.$$