ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{(w_a+w_b+w_c)^2}{h_ah_b+h_bh_c+h_ch_a} \le \frac{1}{12}+\frac{35R}{24r}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a = \sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\le} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}} \le$$

$$\overset{A-G}{\le} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}} = \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}} = \sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$\Rightarrow \frac{(w_a+w_b+w_c)^2}{h_ah_b+h_bh_c+h_ch_a} \le \frac{(s^2+4Rr+r^2)(4R+r)}{2R.\sum_{cyc}\frac{bc.ca}{4R^2}} = \frac{(s^2+4Rr+r^2)(4R+r)}{2R.\frac{4Rrs}{4R^2}.2s}$$

$$\overset{CBS}{\le} \frac{35R+2r}{24r} \Leftrightarrow (11R-4r)s^2 \overset{?}{\ge}_{①} r(96R^2+48Rr+6r^2)$$

$$\text{Now, } (11R-4r)s^2 \overset{Gerretsen}{\ge} (11R-4r)(16Rr-5r^2) \overset{?}{\ge} r(96R^2+48Rr+6r^2)$$

$$\Leftrightarrow 80R^2-167Rr+14r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(80R-7r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r$$

$$\Rightarrow ① \text{ is true} \therefore \frac{(w_a+w_b+w_c)^2}{h_ah_b+h_bh_c+h_ch_a} \le \frac{1}{12}+\frac{35R}{24r}\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)