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In any Δ ABC, the following relationship holds :

$$(R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s)\ge\frac{10\sqrt{3}-9}{3}\cdot F$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s)\ge\frac{10\sqrt{3}-9}{3}.F$$

$$\Leftrightarrow s^2+R^2+r^2-Rr-(R+r)s+3rs\ge\frac{10}{\sqrt{3}}.rs$$

$$\Leftrightarrow s^2+R^2+r^2-Rr\ge(R-2r)s+\frac{10}{\sqrt{3}}.rs$$

$$\Leftrightarrow (s^2+R^2+r^2-Rr)^2\ge(R-2r)^2s^2+\frac{100r^2s^2}{3}+\frac{20}{\sqrt{3}}.r(R-2r)s^2 \text{ and}$$

$$\because\frac{20}{\sqrt{3}}<\frac{35}{3}\ \therefore \text{ it suffices to prove :}$$

$$(s^2+R^2+r^2-Rr)^2\ge(R-2r)^2s^2+\frac{100r^2s^2}{3}+\frac{35}{3}r(R-2r)s^2$$

$$\Leftrightarrow 3s^4+(3R^2-29Rr-36r^2)s^2+3R^4-6R^3r+9R^2r^2-6Rr^3+3r^4\overset{①}{\ge}0$$

Now, $\xi=3(s^2-16Rr+5r^2)^2+(3R^2+67Rr-66r^2)(s^2-16Rr+5r^2)\ge$ $\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove ①, it suffices to prove : LHS of ① $\ge\xi$

$$\Leftrightarrow 3t^4+42t^3+298t^2-917t+258\ge0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t^3+48t^2+394t-129)\ge0\rightarrow\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow ①\text{ is true}$$

$$\therefore (R-s)(R-r)+(s-r)(s-R)+(r-R)(r-s)\ge\frac{10\sqrt{3}-9}{3}.F\ \forall\Delta\text{ ABC},$$

$$''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$