ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$(s-R)(R-r)(s-r) \geq \left(29-9\sqrt{3}\right) \cdot \sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3}$$

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$$(s-R)(R-r)(s-r) \overset{Walker}{\geq}$$

$$\geq \left(\sqrt{2R^2+8Rr+3r^2}-R\right)(R-r)\left(\sqrt{2R^2+8Rr+3r^2}-r\right) \to (1) \text{ and}$$

$$\frac{rs}{3\sqrt{3}} \overset{Mitrinovic}{\leq} \frac{Rr}{2} \Rightarrow \left(29-9\sqrt{3}\right).\sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3} \leq \left(29-9\sqrt{3}\right).\left(\frac{Rr}{2}\right)^{\frac{3}{2}} \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \text{it suffices to prove :}$$

$$\left(\sqrt{2R^2+8Rr+3r^2}-R\right)(R-r)\left(\sqrt{2R^2+8Rr+3r^2}-r\right) \geq \left(29-9\sqrt{3}\right).\left(\frac{Rr}{2}\right)^{\frac{3}{2}}$$

$$\Leftrightarrow \left(\sqrt{2t^2+8t+3}-t\right)(t-1)\left(\sqrt{2t^2+8t+3}-1\right) \overset{①}{\geq} \left(29-9\sqrt{3}\right).\left(\frac{t}{2}\right)^{\frac{3}{2}} \left(t=\frac{R}{r}\right)$$

$$\text{Let } f(t) = \left(\sqrt{2t^2+8t+3}-t\right)(t-1)\left(\sqrt{2t^2+8t+3}-1\right) - \left(29-9\sqrt{3}\right).\left(\frac{t}{2}\right)^{\frac{3}{2}}$$

$$\forall\, t \in [2,\infty) \text{ and then :}$$

$$f'(t) = 6t^2+8t-6-\frac{6t^3+20t^2+4t-4}{\sqrt{2t^2+8t+3}}+6t+\frac{\left(3^{\frac{7}{2}}-87\right).\sqrt{t}}{2^{\frac{5}{2}}}$$

$$\overset{Euler}{\geq} 6t^2+8t-6-\frac{6t^3+20t^2+4t-4}{\sqrt{2t^2+8t+3}}+\left(6\sqrt{2}+\frac{3^{\frac{7}{2}}-87}{4\sqrt{2}}\right).\sqrt{t}$$

$$> \frac{\sqrt{2t^2+8t+3}.(6t^2+8t-6)-(6t^3+20t^2+4t-4)}{\sqrt{2t^2+8t+3}}$$

$$= \frac{2\left(9t^6+60t^5+103t^4-20t^3-144t^2+8t+23\right)}{\sqrt{2t^2+8t+3}.\left(\sqrt{2t^2+8t+3}.(6t^2+8t-6)+(6t^3+20t^2+4t-4)\right)}$$

$$= \frac{2\left(9t^6+60t^5+57t^4+10t^3(t-2)+36t^2(t^2-4)+8t+23\right)}{\sqrt{2t^2+8t+3}.\left(\sqrt{2t^2+8t+3}.(6t^2+8t-6)+(6t^3+20t^2+4t-4)\right)} \overset{Euler}{>} 0$$

$$\Rightarrow f(t) \text{ is } \uparrow \text{ on } [2,\infty) \Rightarrow f(t) \geq f(2) = 0 \Rightarrow ① \text{ is true}$$

$$\therefore (s-R)(R-r)(s-r) \geq \left(29-9\sqrt{3}\right).\sqrt{\left(\frac{F}{3\sqrt{3}}\right)^3}\ \forall \text{ acute } \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$