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In any Δ ABC, the following relationship holds :

$$3\sqrt{3r^3} \le h_a\sqrt{r_a} \le \frac{2s^2}{3\sqrt{6R}}$$

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$$h_a.\sqrt{r_a} \le \frac{2s^2}{3.\sqrt{6R}} \Leftrightarrow \left(\frac{2rs}{a}.\sqrt{\frac{rs}{s-a}}\right)^2 \le \left(\frac{2s^2}{3.\sqrt{6R}}\right)^2 \Leftrightarrow \frac{4r^2s^2}{a^2}.\frac{rs}{s-a} \le \frac{4s^4}{54R}$$

$$\Leftrightarrow a^2bc.\frac{s(s-a)}{bc} \ge 54Rr^3 \Leftrightarrow 4Rrs.a.\cos^2\frac{A}{2} \ge 54Rr^3 \Leftrightarrow 2s^2a\cos^2\frac{A}{2} \overset{(*)}{\ge} 27r^2s$$

$$\text{Now, } 2s^2.a\cos^2\frac{A}{2} \overset{\text{Gerretsen + Euler}}{\ge} 27Rr.a\cos^2\frac{A}{2} \overset{?}{\ge} 27r^2s \Leftrightarrow \frac{R}{r}.a \overset{?}{\ge} s\sec^2\frac{A}{2}$$

$$\Leftrightarrow \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.4R\sin\frac{A}{2}\cos\frac{A}{2} \overset{?}{\ge} \frac{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^2\frac{A}{2}}$$

$$\Leftrightarrow \cos^2\frac{A}{2} \overset{?}{\ge} \left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$\Leftrightarrow \cos^2\frac{A}{2} \overset{?}{\underset{(**)}{\ge}} \left(\sin\frac{A}{2} + \cos\frac{B-C}{2}\right)\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)$$

$$\text{Now, } \because \cos\frac{B-C}{2} \le 1 \therefore \left(\sin\frac{A}{2} + \cos\frac{B-C}{2}\right)\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) \le$$

$$\left(1+\sin\frac{A}{2}\right)\left(1-\sin\frac{A}{2}\right) = 1 - \sin^2\frac{A}{2} = \cos^2\frac{A}{2} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore h_a.\sqrt{r_a} \le \frac{2s^2}{3.\sqrt{6R}}$$

$$\text{Again, } h_a.\sqrt{r_a} \ge 3.\sqrt{3r^3} \Leftrightarrow \left(\frac{2rs}{a}.\sqrt{\frac{rs}{s-a}}\right)^2 \ge 3.\sqrt{3r^3} \Leftrightarrow \frac{4s^3}{a^2(s-a)} \ge 27$$

$$\Leftrightarrow 8s^3 \ge 27a^2(b+c-a) \to \text{true} \because \sqrt[3]{a^2(b+c-a)} \overset{A-G}{\le} \frac{a+a+b+c-a}{3} = \frac{2s}{3}$$

$$\Rightarrow a^2(b+c-a) \le \frac{8s^3}{27} \therefore h_a.\sqrt{r_a} \ge 3.\sqrt{3r^3} \text{ and so, } \forall\, \Delta \text{ ABC}, h_a.\sqrt{r_a} \le \frac{2s^2}{3.\sqrt{6R}},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral and } h_a.\sqrt{r_a} \ge 3.\sqrt{3r^3}, '' ='' \text{ iff } b+c = 2a \text{ (QED)}$$