ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2}\le\frac{2(R+r)}{R\sqrt{3r}}$$

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$$\text{Firstly, }\sum_{cyc}\left((s-b)(s-c)(b+c)^2\right)=\sum_{cyc}\left((-s^2+sa+bc)(b^2+c^2+2bc)\right)=$$

$$=-2s^2\sum_{cyc}a^2-2s^2\sum_{cyc}ab+s\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)+6sabc+$$

$$+\sum_{cyc}\left(bc\left(\sum_{cyc}a^2-a^2\right)\right)+2\sum_{cyc}a^2b^2$$

$$=-4s^2(s^2-4Rr-r^2)+12Rrs^2+2(s^2-4Rr-r^2)(s^2+4Rr+r^2)-8Rrs^2$$

$$+2(s^2+4Rr+r^2)^2-32Rrs^2=4r(R+2r)s^2\overset{(i)}{=}\sum_{cyc}\left((s-b)(s-c)(b+c)^2\right)$$

$$\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)=\sum_{cyc}\left(bc(s-b)(s-c)\left(\sum_{cyc}a^2-a^2+2bc\right)\right)=$$

$$=\left(\sum_{cyc}a^2\right)r^2s^2\sum_{cyc}\frac{bc}{s(s-a)}-4Rrs.\sum_{cyc}\left(a(-s^2+sa+bc)\right)+2\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)=$$

$$=2r^2s^2(s^2-4Rr-r^2).\frac{s^2+(4R+r)^2}{s^2}$$

$$-4Rrs.(-s^2(2s)+2s(s^2-4Rr-r^2)+12Rrs)-2s^2((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$+8Rrs^2(s^2+4Rr+r^2)+2(s^2+4Rr+r^2)^3-48Rrs^2(s^2+2Rr+r^2)$$

$$=4r^2s^2(4R^2+2Rr+r^2+s^2)\overset{(ii)}{=}\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)$$

$$\text{Now, }\frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2}=\sum_{cyc}\left(\frac{bc}{2R}.\sqrt{\frac{bc}{2R}}.\frac{(b+c)^2}{4bc}.\frac{(s-b)(s-c)}{r^2s^2}\right)$$

$$=\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sum_{cyc}\left(\sqrt{bc}.(s-b)(s-c)(b+c)^2\right)$$

$$= \frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sum_{cyc}\left(\sqrt{(s-b)(s-c)(b+c)^2}.\sqrt{bc(s-b)(s-c)(b+c)^2}\right) \overset{CBS}{\leq}$$

$$\frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sqrt{\sum_{cyc}((s-b)(s-c)(b+c)^2)}.\sqrt{\sum_{cyc}(bc(s-b)(s-c)(b+c)^2)}$$

$$\overset{via\ (i)\ and\ (ii)}{=} \frac{1}{2R.\sqrt{2R}.4r^2s^2}.\sqrt{4r(R+2r)s^2}.\sqrt{4r^2s^2(4R^2+2Rr+r^2+s^2)} =$$

$$\frac{1}{2R}.\sqrt{\frac{(R+2r)(4R^2+2Rr+r^2+s^2)}{2Rr}} \overset{Gerretsen}{\leq} \frac{1}{2R}.\sqrt{\frac{(R+2r)(8R^2+6Rr+4r^2)}{2Rr}} \overset{?}{\leq} \frac{2(R+r)}{R.\sqrt{3r}}$$

$$\Leftrightarrow 4t^3-t^2-8t-12 \overset{?}{\geq} 0\ \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(4t^2+7t+6) \overset{?}{\geq} 0 \to true \because t \overset{Euler}{\geq} 2$$

$$\therefore \frac{h_a\sqrt{h_a}}{w_a^2}+\frac{h_b\sqrt{h_b}}{w_b^2}+\frac{h_c\sqrt{h_c}}{w_c^2} \leq \frac{2(R+r)}{R.\sqrt{3r}}\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)