ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$(w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)$$

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$$2h_ar_a=\frac{4rs^2\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=\frac{rs^2}{R\cos^2\frac{A}{2}}\Rightarrow\frac{1}{\sqrt{h_ar_a}}=\frac{1}{s}.\sqrt{\frac{2R}{r}}\cos\frac{A}{2}\text{ and analogs}$$

$$\Rightarrow 3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)=3\sum_{cyc}\cos\frac{A}{2}\to\text{(i)}$$

$$\text{Now, } w_a+w_b+w_c=\sum_{cyc}\left(\frac{2bc}{b+c}.\cos\frac{A}{2}\right)\overset{\text{Chebyshev}}{\ge}\frac{2}{3}.\sum_{cyc}\frac{bc}{b+c}.\sum_{cyc}\cos\frac{A}{2}$$

$$\left(\because\text{ WLOG assuming } a\ge b\ge c\Rightarrow\frac{bc}{b+c}\le\frac{ca}{c+a}\le\frac{ab}{a+b}\text{ and } \cos\frac{A}{2}\le\cos\frac{B}{2}\le\cos\frac{C}{2}\right)$$

$$=\frac{2}{3}.\frac{1}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(bc\left(a^2+\sum_{cyc}ab\right)\right).\sum_{cyc}\cos\frac{A}{2}$$

$$=\frac{(s^2+4Rr+r^2)^2+8Rrs^2}{3s(s^2+2Rr+r^2)}.\sum_{cyc}\cos\frac{A}{2}\Rightarrow(w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge$$

$$\frac{((s^2+4Rr+r^2)^2+8Rrs^2)(s^2+4Rr+r^2)}{3s(s^2+2Rr+r^2).4Rrs}.\sum_{cyc}\cos\frac{A}{2}\overset{?}{\ge}$$

$$3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_ar_a}}+\frac{1}{\sqrt{h_br_b}}+\frac{1}{\sqrt{h_cr_c}}\right)\overset{\text{via (i)}}{=}3\sum_{cyc}\cos\frac{A}{2}$$

$$\Leftrightarrow((s^2+4Rr+r^2)^2+8Rrs^2)(s^2+4Rr+r^2)\overset{?}{\ge}36Rrs^2(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^6-(16Rr-3r^2)s^4+r^2(8R^2-4Rr+3r^2)s^2+r^3(4R+r)^3\ \boxed{\overset{?}{\underset{①}{\ge}}}\ 0\text{ and }\because$$

$(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①}\ge(s^2-16Rr+5r^2)^3\Leftrightarrow(8R-3r)s^4-r(190R^2-119Rr+18r^2)s^2$$

$$+r^2(1040R^3-948R^2r+303Rr^2-31r^3)\ \boxed{\overset{②}{\ge}}\ 0$$

and $\because(8R-3r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove ②,

$$\text{it suffices to prove : LHS of ②} \geq (8R-3r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (66R^2-57Rr+12r^2)s^2 \overset{③}{\geq} r(1008R^3-1100R^2r+377Rr^2-44r^3) \text{ and}$$

$$\text{finally, } (66R^2-57Rr+12r^2)s^2 \overset{\text{Gerretsen}}{\geq} (66R^2-57Rr+12r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(1008R^3-1100R^2r+377Rr^2-44r^3) \Leftrightarrow 24t^3-71t^2+50t-8 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(12t^2+12t(t-2)+t+4) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ② \Rightarrow ①$$

$$\text{is true} \therefore (w_a+w_b+w_c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq$$

$$3s.\sqrt{\frac{r}{2R}}\left(\frac{1}{\sqrt{h_a r_a}}+\frac{1}{\sqrt{h_b r_b}}+\frac{1}{\sqrt{h_c r_c}}\right) \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$