ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c \le h_a w_b w_c + w_a h_b w_c + w_a w_b h_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c$$

$$\Leftrightarrow \sum_{cyc} \frac{w_a}{h_a} \le \frac{16Rr^2s^2}{s^2+2Rr+r^2} \cdot \frac{R}{2r^2s^2} + 2$$

$$\Leftrightarrow \sum_{cyc} \left(\frac{2bc\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}} \cdot \frac{2R}{bc} \right) \le \frac{8R^2}{s^2+2Rr+r^2} + 2$$

$$\Leftrightarrow \frac{1}{\prod_{cyc}\cos\frac{B-C}{2}} \cdot \sum_{cyc} \left(\cos\frac{C-A}{2}\cos\frac{A-B}{2} \right) \overset{①}{\le} \frac{8R^2}{s^2+2Rr+r^2} + 2$$

$$\text{Now, } \cos\frac{B-C}{2} + \cos\frac{B+C}{2} = 2\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2s}{4R}\sec\frac{A}{2} \Rightarrow$$

$$\cos^2\frac{B-C}{2} = \left(\frac{s}{2R}\sec\frac{A}{2} - \sin\frac{A}{2} \right)^2 \Rightarrow$$

$$\cos^2\frac{B-C}{2} = \frac{s^2}{4R^2}\sec^2\frac{A}{2} + \sin^2\frac{A}{2} - \frac{s}{R}\tan\frac{A}{2} \text{ and analogs}$$

$$\therefore \sum_{cyc}\cos^2\frac{B-C}{2} = \frac{s^2}{4R^2} \cdot \sum_{cyc}\sec^2\frac{A}{2} + \sum_{cyc}\sin^2\frac{A}{2} - \frac{1}{R}\sum_{cyc} s\tan\frac{A}{2}$$

$$= \frac{s^2}{4R^2} \cdot \frac{s^2+(4R+r)^2}{s^2} + \frac{2R-r}{2R} - \frac{4R+r}{R}$$

$$= \frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow \sum_{cyc}\cos^2\frac{B-C}{2} = \frac{s^2+4R^2+2Rr+r^2}{4R^2} \to (m) \text{ and}$$

$$\prod_{cyc}\cos\frac{B-C}{2} = \prod_{cyc}\frac{b+c}{a} \cdot \prod_{cyc}\sin\frac{A}{2} = \frac{2s(s^2+2Rr+r^2)}{4Rrs} \cdot \frac{r}{4R}$$

$$\Rightarrow \prod_{cyc}\cos\frac{B-C}{2} = \frac{s^2+2Rr+r^2}{8R^2} \to (n)$$

$$\text{Now, LHS of } ① \le \frac{1}{\prod_{cyc}\cos\frac{B-C}{2}}\cdot\sum_{cyc}\cos^2\frac{B-C}{2} \overset{\text{via (m) and (n)}}{=}$$
$$\frac{8R^2}{s^2+2Rr+r^2}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R^2} = \frac{2(s^2+2Rr+r^2)+8R^2}{s^2+2Rr+r^2}$$
$$= \frac{8R^2}{s^2+2Rr+r^2} + 2 \Rightarrow ① \text{ is true}$$

$$\therefore w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c$$
$$\text{Again, } w_a w_b w_c + 2h_a h_b h_c \le h_a w_b w_c + w_a h_b w_c + w_a w_b h_c$$

$$\Leftrightarrow \sum_{cyc}\frac{h_a}{w_a} \ge \frac{4r^2s^2}{R}\cdot\frac{s^2+2Rr+r^2}{16Rr^2s^2} + 1$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}}\right) \ge \frac{s^2+2Rr+r^2+4R^2}{4R^2}$$
$$\Leftrightarrow \sum_{cyc}\cos^2\frac{B-C}{2} + 2\sum_{cyc}\left(\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right) \ge \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$
$$\overset{\text{via (m) and (n)}}{\Leftrightarrow} \frac{s^2+4R^2+2Rr+r^2}{4R^2} + \frac{s^2+2Rr+r^2}{4R^2}\cdot\sum_{cyc}\frac{1}{\cos\frac{B-C}{2}}$$
$$\overset{②}{\ge} \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$

$$\text{Now, } \because 0 < \cos\frac{B-C}{2} \le 1 \therefore \text{LHS of } ② \ge \frac{s^2+4R^2+2Rr+r^2}{4R^2} + \frac{3s^2+6Rr+3r^2}{4R^2}$$
$$= \frac{s^2+R^2+2Rr+r^2}{R^2} \overset{?}{\ge} \frac{(s^2+4R^2+2Rr+r^2)^2}{16R^4}$$

$$\Leftrightarrow \frac{s^2+2Rr+r^2}{R^2} + 1 \overset{?}{\ge} \frac{(s^2+2Rr+r^2)^2}{16R^4} + \frac{8R^2(s^2+2Rr+r^2)}{16R^4} + 1$$

$$\Leftrightarrow \frac{1}{2R^2} \overset{?}{\ge} \frac{s^2+2Rr+r^2}{16R^4} \Leftrightarrow 8R^2 - 2Rr - r^2 \overset{?}{\ge} s^2 \to \text{true}$$

$$\because 8R^2 - 2Rr - r^2 = 4R^2 + 4Rr + 3r^2 + 2(R-2r)(2R+r) \overset{\text{Gerretsen and Euler}}{\ge} s^2$$

$\Rightarrow ②$ is true $\therefore w_a w_b w_c + 2h_a h_b h_c \le h_a w_b w_c + w_a h_b w_c + w_a w_b h_c$ *and* so,
$w_a h_b h_c + h_a w_b h_c + h_a h_b w_c \le w_a w_b w_c + 2h_a h_b h_c \le$
$h_a w_b w_c + w_a h_b w_c + w_a w_b h_c \ \forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)