ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\frac{R}{s+r}+\frac{s}{r+R}+\frac{r}{R+s}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s+r}+\frac{s}{r+R}+\frac{r}{R+s}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\Leftrightarrow (R+r+s)\left(\frac{1}{s+r}+\frac{1}{r+R}+\frac{1}{R+s}\right)>2s\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$$\Leftrightarrow \frac{(R+r+s)(s^2+3s(R+r)+R^2+3Rr+r^2)}{(R+r)(s^2+s(R+r)+Rr)}>\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}$$

$$\Leftrightarrow s^5-(R+r)s^4-(R^2-Rr)s^3+(R^3+3R^2r+6Rr^2+4r^3)s^2$$
$$+r(R^3+13R^2r+11Rr^2+3r^3)s+r(2R^4+5R^3r+7R^2r^2+5Rr^3+r^4)\overset{①}{>}0$$

Now, $\because \Delta$ ABC is acute $\therefore s>2R+r$ and so:

$$P=(s-2R-r)^5+(9R+4r)(s-2R-r)^4+(31R^2+29Rr+6r^2)(s-2R-r)^3+$$
$$+(51R^3+78R^2r+39Rr^2+8r^3)(s-2R-r)^2$$
$$+2(20R^4+49R^3r+50R^2r^2+27Rr^3+6r^4)(s-2R-r)>0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $>P$

$$\Leftrightarrow 6R^5+27R^4r+51R^3r^2+49R^2r^3+23Rr^4+4r^5>0\to \text{true}$$

$$\therefore \frac{R}{s+r}+\frac{s}{r+R}+\frac{r}{R+s}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ \forall\,\Delta\,ABC\ (QED)$$