In *any* △ ABC, the fo*llow*ing re*la*tionship ho*l*ds :

$$h_a + w_a + m_a - r_a - r_b - r_c \le 4\sqrt{R(R - 2r)}$$

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$$w_a = \frac{2bc}{b + c} \cdot \cos\frac{A}{2} = \frac{4R^2(\cos(B - C) + \cos A)\cdot\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B - C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B - C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B - C}{2}} = \frac{2R(c^2 - s^2)}{c} \quad \left(c = \cos\frac{B - C}{2} \ and \ s = \sin\frac{A}{2}\right)$$

$$\overset{?}{\le} R + r + \sqrt{R(R - 2r)}$$

$$= R + 2R\sin\frac{A}{2}\left(\cos\frac{B - C}{2} - \sin\frac{A}{2}\right) + R\cdot\sqrt{1 - 4sc + 4s^2}$$

$$\Leftrightarrow 2c - \frac{2s^2}{c} \overset{?}{\le} 1 + 2sc - 2s^2 + \sqrt{1 - 4sc + 4s^2}$$

$$\Leftrightarrow 1 + 2sc - 2c + \frac{2s^2}{c} - 2s^2 + \sqrt{1 - 4sc + 4s^2} \boxed{\overset{?}{\underset{①}{\ge}}} 0$$

Now, $\dfrac{2s^2}{c} - 2s^2 = \dfrac{2s^2\left(1 - \cos\frac{B - C}{2}\right)}{\cos\frac{B - C}{2}} \ge 0$ and $1 - 4sc + 4s^2 \overset{0 < c \le 1}{\ge} 1 - 4s + 4s^2$

$$= (1 - 2s)^2 \Rightarrow \sqrt{1 - 4sc + 4s^2} \ge |1 - 2s| \ and \ so, in \ order \ to \ prove \ ①,$$

it suffices to prove : $1 + 2sc - 2c + |1 - 2s| \boxed{\overset{②}{\ge}} 0$

$\boxed{Case \ 1}$ $1 - 2s \ge 0$ *and then* : LHS of ② $= 1 + 2sc - 2c + 1 - 2s$

$$= 1 - c - c(1 - 2s) + 1 - 2s = (1 - 2s)(1 - c) + (1 - c) = 2(1 - c)(1 - s) \ge 0$$

$$\because c = \cos\frac{B - C}{2} \le 1 \ and \ s = \sin\frac{A}{2} < 1 \Rightarrow ② \ is \ true$$

$\boxed{Case \ 2}$ $1 - 2s < 0$ *and then* : LHS of ② $= 1 + 2sc - 2c + 2s - 1$

$$= 1 - c + c(2s - 1) + (2s - 1) \overset{c \le 1}{\ge} (2s - 1)(1 + c) > 0 \because 1 - 2s < 0$$

$$\Rightarrow ② \ is \ true \ (strict \ inequ\textit{a}lity) \therefore combining \ both \ c\textit{a}ses,$$

② is true $\forall$ △ ABC $\therefore$ $\boxed{w_a \le R + r + \sqrt{R(R - 2r)}}$ $\forall$ △ ABC $\rightarrow$ (m)

We sh*all* now prove th*at* : $m_a \le 2R - r + 2\cdot\sqrt{R(R - 2r)}$ $\forall$ △ ABC

$\boxed{Case \ 1}$ $\widehat{A}$ *is* acute *and then* : $m_a \le 2R\cos^2\frac{A}{2} \overset{?}{\le} 2R - r + 2\cdot\sqrt{R(R - 2r)}$

$$\Leftrightarrow 2Rs^2 - 2Rs(c - s) + 2R\cdot\sqrt{1 - 4sc + 4s^2} \overset{?}{\ge} 0 \Leftrightarrow \sqrt{1 - 4sc + 4s^2} \boxed{\overset{?}{\underset{③}{\ge}}} sc - 2s^2$$

which is trivi*all*y true if $sc - 2s^2 < 0$ *and so*, we now focus on the scen*a*rio

when $:sc - 2s^2 \geq 0$ and then $:$ ③ $\Leftrightarrow 1 - 4sc + 4s^2 \overset{?}{\underset{④}{\geq}} s^2c^2 + 4s^4 - 4cs^3$ and

$\because c \leq 1 \therefore$ in order to prove ④, it suffices to prove $:$

$1 - 4sc + 4s^2 \overset{?}{\geq} s^2 + 4s^4 - 4cs^3 \Leftrightarrow 1 - s^2 + 4s^2(1 - s^2) - 4sc(1 - s^2) \overset{?}{\geq} 0$

$\Leftrightarrow (1 - s^2)(1 - 4sc + 4s^2) \boxed{\overset{?}{\geq}} 0 \Leftrightarrow \cos^2 \dfrac{A}{2} \cdot \dfrac{R - 2r}{R} \overset{?}{\geq} 0 \to$ true

$\Rightarrow$ ④ $\Rightarrow$ ③ is true $\therefore m_a \leq 2R - r + 2.\sqrt{R(R - 2r)}$

$\boxed{Case\ 2}\ \widehat{A} \geq \dfrac{\pi}{2}$ and then $:4m_a^2 = 2b^2 + 2c^2 - 2a^2 + a^2 = 4bc\cos A + a^2 \leq a^2$

$\Rightarrow m_a \leq \dfrac{a}{2} = R\sin A \leq R \overset{?}{\leq} 2R - r + 2.\sqrt{R(R - 2r)} \Leftrightarrow R - r + 2.\sqrt{R(R - 2r)} \overset{?}{\geq} 0$

$\to$ true (strict inequality) $\therefore$ combining both cases,

$\boxed{m_a \leq 2R - r + 2.\sqrt{R(R - 2r)}}\ \forall\ \Delta\ ABC \to$ (n)

So, $h_a + w_a + m_a - r_a - r_b - r_c \leq 2w_a + m_a - 4R - r \overset{via\ (m)\ and\ (n)}{\leq}$

$2R + 2r + 2\sqrt{R(R - 2r)} + 2R - r + 2.\sqrt{R(R - 2r)} - 4R - r = 4.\sqrt{R(R - 2r)}$

$\therefore h_a + w_a + m_a - r_a - r_b - r_c \leq 4.\sqrt{R(R - 2r)}\ \forall\ \Delta\ ABC,$

$'' = ''$ iff $\Delta\ ABC$ is equilateral (QED)