ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{w_a w_b w_c}+\frac{1}{w_a w_b h_c}+\frac{1}{w_a h_b w_c}+\frac{1}{h_a w_b w_c}\geq\frac{2(R+2r)}{s^2Rr}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$w_a w_b w_c\overset{w_a\leq\sqrt{s(s-a)}}{\leq}\sqrt{s(s-a)s(s-b)s(s-c)}\leq\sqrt{s^4r^2}=s^2r\ (1)$$

We need to show:

$$\frac{1}{w_a w_b w_c}+\frac{1}{w_a w_b h_c}+\frac{1}{w_a h_b w_c}+\frac{1}{h_a w_b w_c}\geq\frac{2(R+2r)}{s^2Rr}$$

$$or,\frac{s^2r}{w_a w_b w_c}+\frac{s^2r}{w_a w_b h_c}+\frac{s^2r}{w_a h_b w_c}+\frac{s^2r}{h_a w_b w_c}\geq\frac{2(R+2r)}{R}$$

$$\frac{w_a w_b w_c}{w_a w_b w_c}+\frac{w_a w_b w_c}{w_a w_b h_c}+\frac{w_a w_b w_c}{w_a h_b w_c}+\frac{w_a w_b w_c}{h_a w_b w_c}\geq\frac{2(R+2r)}{R}\ (using\ (1))$$

$$1+\frac{w_c}{h_c}+\frac{w_b}{h_b}+\frac{w_a}{h_a}\overset{w_a\geq h_a\ or,\frac{w_a}{h_a}\geq 1}{\geq}\frac{2(R+2r)}{R}$$

$$1+1+1+1\geq\frac{2(R+2r)}{R}$$

$$4R\geq 2R+4r\ or\ R\geq 2r\ Euler$$

$$Equality\ holds\ for\ a=b=c.$$