ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$w_a + w_b + w_c \le s^2 r\left(\frac{1}{w_a\sqrt{r_a h_a}} + \frac{1}{w_b\sqrt{r_b h_b}} + \frac{1}{w_c\sqrt{r_c h_c}}\right)$$

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$$\sum w_a = \sum \frac{2bc}{b+c}\cos\frac{A}{2} \overset{AM-GM}{\le} \sum \frac{2bc}{2\sqrt{bc}}\cos\frac{A}{2} =$$

$$= \sum \sqrt{bc}\cos\frac{A}{2} \overset{CBS}{\le} \sqrt{\left(\sum bc\right)\left(\sum \cos^2\frac{A}{2}\right)} = \sqrt{\left(\sum bc\right)\left(\frac{4R+r}{2R}\right)} \ (1)$$

$$w_a\sqrt{r_a h_a} = \frac{2\sqrt{bcs(s-a)}}{b+c}\cdot\sqrt{\frac{rs}{s-a}\cdot\frac{2rs}{a}} = \frac{2\sqrt{2}rs\sqrt{bcs}}{(b+c)\sqrt{a}} = \frac{2\sqrt{2}rs\sqrt{abcs}}{(b+ç)a} =$$

$$= \frac{2\sqrt{2}rs\sqrt{4Rrs^2}}{(ab+aç)} = \frac{2\sqrt{2}s^2r\sqrt{4Rr}}{ab+ac} = \frac{2s^2r\sqrt{8Rr}}{ab+ac} \ (2)$$

$$s^2r\left(\frac{1}{w_a\sqrt{r_a h_a}} + \frac{1}{w_b\sqrt{r_b h_b}} + \frac{1}{w_c\sqrt{r_c h_c}}\right) = s^2r\sum\frac{1}{w_a\sqrt{r_a h_a}} \overset{(2)}{\ge}$$

$$\ge s^2r\sum\frac{ab+ac}{2s^2r\sqrt{8Rr}} = \frac{2\sum ab}{2\sqrt{8Rr}} \ (3)$$

From $(1)\&(3)$ *we need to show*:

$$\frac{2\sum ab}{2\sqrt{8Rr}} \ge \sqrt{\left(\sum bc\right)\left(\frac{4R+r}{2R}\right)} \ \text{or} \ \sqrt{\sum ab} \ge \sqrt{8Rr\left(\frac{4R+r}{2R}\right)}$$

$$\left(\sqrt{\sum ab}\right)^2 \ge \left(\sqrt{8Rr\left(\frac{4R+r}{2R}\right)}\right)^2 \ \text{or} \sum ab \ge 4r(4R+r)$$

$$\text{or}, s^2 + r^2 + 4Rr \ge 16Rr + 4r^2$$

$$16Rr - 5r^2 + r^2 + 4Rr \ge 16Rr + 4r^2 (Gerretsen)$$

$$4Rr \ge 8r^2 \ \text{or}, R \ge 2r \ (Euler) \ true$$

Equality holds for $a = b = c$