ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right)\ge\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}\ge 1+\frac{9r}{2R}-\frac{r^2}{R^2}$$

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$$\sum_{cyc}\frac{h_a}{w_a}=\sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}}\right)\therefore\sum_{cyc}\frac{h_a}{w_a}=\sum_{cyc}\cos\frac{B-C}{2}\to(m)$$

$$\text{Now, }\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}R-\frac{r}{2}\right)=\frac{1}{\sqrt{2Rr}}\cdot\left(\sum_{cyc}\frac{s(s-a)}{r_b+r_c}+\frac{2R-r}{2}\right)$$

$$=\frac{1}{\sqrt{2Rr}}\cdot\left(\sum_{cyc}\frac{bc\cos^2\frac{A}{2}}{4R\cos^2\frac{A}{2}}+\frac{2R-r}{2}\right)=\frac{1}{\sqrt{2Rr}}\cdot\left(\frac{s^2+4Rr+r^2}{4R}+\frac{2R-r}{2}\right)$$

$$\therefore\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}R-\frac{r}{2}\right)=\frac{1}{\sqrt{2Rr}}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R}\to(n)$$

$$\text{Now, }\cos\frac{B-C}{2}+\cos\frac{B+C}{2}=2\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2s}{4R}\sec\frac{A}{2}\Rightarrow\cos^2\frac{B-C}{2}$$

$$=\left(\frac{s}{2R}\sec\frac{A}{2}-\sin\frac{A}{2}\right)^2\Rightarrow\cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\sec^2\frac{A}{2}+\sin^2\frac{A}{2}-\frac{s}{R}\tan\frac{A}{2}\text{ and analogs}$$

$$\therefore\sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\cdot\sum_{cyc}\sec^2\frac{A}{2}+\sum_{cyc}\sin^2\frac{A}{2}-\frac{1}{R}\sum_{cyc}s\tan\frac{A}{2}$$

$$=\frac{s^2}{4R^2}\cdot\frac{s^2+(4R+r)^2}{s^2}+\frac{2R-r}{2R}-\frac{4R+r}{R}$$

$$=\frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow\sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2+4R^2+2Rr+r^2}{4R^2}\to(r)\text{ and }\prod_{cyc}\cos\frac{B-C}{2}=$$

$$\prod_{cyc}\frac{b+c}{a}\cdot\prod_{cyc}\sin\frac{A}{2}=\frac{2s(s^2+2Rr+r^2)}{4Rrs}\cdot\frac{r}{4R}\Rightarrow\prod_{cyc}\cos\frac{B-C}{2}\overset{(s)}{=}\frac{s^2+2Rr+r^2}{8R^2}$$

$$\text{We have : }\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}\overset{\text{via (m)}}{=}\sum_{cyc}\cos\frac{B-C}{2}\overset{CBS}{\le}\sqrt{3\sum_{cyc}\cos^2\frac{B-C}{2}}\overset{\text{via (r)}}{=}$$

$$\sqrt{\frac{3(s^2+4R^2+2Rr+r^2)}{4R^2}} \overset{?}{\leq} \frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right)$$

$$\overset{\text{via (n)}}{=} \frac{1}{\sqrt{2Rr}}\cdot\frac{s^2+4R^2+2Rr+r^2}{4R} \Leftrightarrow s^2+4R^2+2Rr+r^2 \underset{①}{\overset{?}{\geq}} 24Rr$$

Via Gerretsen, LHS of ① $\geq 4R^2+18Rr-4r^2 \overset{?}{\geq} 24Rr \Leftrightarrow 2R^2-3Rr-2r^2 \overset{?}{\geq} 0$

$\Leftrightarrow (2R-r)(R-2r) \overset{?}{\geq} 0 \to$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow$ ① is true

$$\therefore \frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right) \geq \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}$$

Again, $\left(\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c}\right)^2 \overset{\text{via (m)}}{=} \sum_{cyc}\cos^2\frac{B-C}{2}+2\sum_{cyc}\left(\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)$

$$\overset{\text{via (r)}}{=} \frac{s^2+4R^2+2Rr+r^2}{4R^2}+2\prod_{cyc}\cos\frac{B-C}{2}\cdot\sum_{cyc}\frac{1}{\cos\frac{B-C}{2}} \overset{\text{via (r)}}{\geq}$$

$$\frac{s^2+4R^2+2Rr+r^2}{4R^2}+\frac{s^2+2Rr+r^2}{4R^2}.(1+1+1)$$

$$\left(\because 0<\cos\frac{B-C}{2}\leq 1 \text{ and analogs}\right) = \frac{s^2+R^2+2Rr+r^2}{R^2} \overset{\text{Gerretsen}}{\geq}$$

$$\frac{R^2+18Rr-4r^2}{R^2} \overset{?}{\geq} \left(1+\frac{9r}{2R}-\frac{r^2}{R^2}\right)^2 = \frac{(2R^2+9Rr-2r^2)^2}{4R^4}$$

$$\Leftrightarrow 36t^3-89t^2+36t-4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(27t^2+9t(t-2)+t+2) \overset{?}{\geq} 0$$

$\to$ true $\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c} \geq 1+\frac{9r}{2R}-\frac{r^2}{R^2}$ *and* so,

$$\frac{1}{\sqrt{2Rr}}\left(\frac{1}{\frac{1}{r_a}+\frac{1}{r_b}}+\frac{1}{\frac{1}{r_b}+\frac{1}{r_c}}+\frac{1}{\frac{1}{r_c}+\frac{1}{r_a}}+R-\frac{r}{2}\right) \geq \frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c} \geq$$

$1+\frac{9r}{2R}-\frac{r^2}{R^2}\ \forall\Delta ABC$," $=$ " iff Δ ABC is equi*lateral* (QED)