ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{3\sqrt{6r}}{\sqrt{R}} \le \frac{h_a\cdot\sqrt{h_a}}{w_a}+\frac{h_b\cdot\sqrt{h_b}}{w_b}+\frac{h_c\cdot\sqrt{h_c}}{w_c} \le \frac{2s^2}{3\sqrt{3r}\cdot R}$$

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$$\frac{h_a}{w_a}=\frac{bc}{2R}\cdot\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2bc\cos\frac{A}{2}} \therefore \frac{h_a}{w_a}=\cos\frac{B-C}{2} \text{ and analogs} \to (m)$$

$$\text{Now, } \cos\frac{B-C}{2}+\cos\frac{B+C}{2}=2\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2s}{4R}\sec\frac{A}{2}$$

$$\Rightarrow \cos^2\frac{B-C}{2}=\left(\frac{s}{2R}\sec\frac{A}{2}-\sin\frac{A}{2}\right)^2$$

$$\Rightarrow \cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\sec^2\frac{A}{2}+\sin^2\frac{A}{2}-\frac{s}{R}\tan\frac{A}{2} \text{ and analogs}$$

$$\therefore \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2}{4R^2}\cdot\sum_{cyc}\sec^2\frac{A}{2}+\sum_{cyc}\sin^2\frac{A}{2}-\frac{1}{R}\sum_{cyc}s\tan\frac{A}{2}$$

$$=\frac{s^2}{4R^2}\cdot\frac{s^2+(4R+r)^2}{s^2}+\frac{2R-r}{2R}-\frac{4R+r}{R}$$

$$=\frac{s^2+(4R+r)^2+2R(2R-r)-4R(4R+r)}{4R^2}$$

$$\Rightarrow \sum_{cyc}\cos^2\frac{B-C}{2}=\frac{s^2+4R^2+2Rr+r^2}{4R^2} \to (n)$$

$$\text{We have : } \frac{h_a\cdot\sqrt{h_a}}{w_a}+\frac{h_b\cdot\sqrt{h_b}}{w_b}+\frac{h_c\cdot\sqrt{h_c}}{w_c} \overset{\text{via }(m)}{=} \sum_{cyc}\left(\cos\frac{B-C}{2}\cdot\sqrt{h_a}\right) \overset{CBS}{\le}$$

$$\sqrt{\sum_{cyc}\cos^2\frac{B-C}{2}}\cdot\sqrt{\frac{1}{2R}\cdot\sum_{cyc}ab} \overset{\text{via }(n)}{=} \sqrt{\frac{s^2+4R^2+2Rr+r^2}{4R^2}\cdot\frac{s^2+4Rr+r^2}{2R}} \overset{?}{\le} \frac{2s^2}{3\sqrt{3r}.R}$$

$$\Leftrightarrow 32Rs^4 \overset{?}{\ge} 27r(s^2+4R^2+2Rr+r^2)(s^2+4Rr+r^2)$$

$$\Leftrightarrow (32R-27r)s^4-r(108R^2+162Rr+54r^2)s^2$$

$$-27r^2(16R^3+12R^2r+6Rr^2+r^3) \overset{?}{\underset{①}{\ge}} 0 \text{ and } \because (32R-27r)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①} \ge (32R-27r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (458R^2-673Rr+108r^2)s^2 \overset{②}{\ge} r(4312R^3-5854R^2r+2641Rr^2-324r^3)$$

$$\text{Now, LHS of } ② \overset{\text{Gerretsen}}{\geq} (458R^2 - 673Rr + 108r^2)(16Rr - 5r^2) \overset{?}{\geq}$$

$$r(4312R^3 - 5854R^2r + 2641Rr^2 - 324r^3) \Leftrightarrow 754t^3 - 1801t^2 + 613t - 54 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(754t^2 - 293t + 27) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \frac{h_a \cdot \sqrt{h_a}}{w_a} + \frac{h_b \cdot \sqrt{h_b}}{w_b} + \frac{h_c \cdot \sqrt{h_c}}{w_c} \leq \frac{2s^2}{3\sqrt{3r}.R}$$

$$\text{Again, } \frac{h_a \cdot \sqrt{h_a}}{w_a} + \frac{h_b \cdot \sqrt{h_b}}{w_b} + \frac{h_c \cdot \sqrt{h_c}}{w_c} = \sum_{cyc} \frac{h_a^2}{\sqrt{h_a}.w_a} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^2}{\sqrt{\sum_{cyc} h_a} \cdot \sqrt{\sum_{cyc} w_a^2}} \geq$$

$$\frac{\left(\sum_{cyc} h_a\right).\sqrt{\sum_{cyc} h_a}}{\sqrt{\sum_{cyc} s(s-a)}} \overset{?}{\geq} \frac{3\sqrt{6}r}{\sqrt{R}} \Leftrightarrow \frac{(s^2+4Rr+r^2)^3}{8R^3} \overset{?}{\geq} \frac{s^2.54r^2}{R}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^3 \underset{③}{\overset{?}{\geq}} 432R^2r^2s^2$$

$$\text{Now, } (s^2+4Rr+r^2)^2 \overset{?}{\geq} 24Rrs^2 \Leftrightarrow s^4 - (16Rr - 2r^2)s^2 + r^2(4R+r)^2 \underset{(i)}{\overset{?}{\geq}} 0$$

$$\text{We have : LHS of (i)} \overset{\text{Gerretsen}}{\geq} -3r^2s^2 + r^2(4R+r)^2 = r^2((4R+r)^2 - 3s^2)$$

$$\overset{\text{Doucet or Trucht}}{\geq} 0 \Rightarrow \text{(i) is true} \Rightarrow (s^2+4Rr+r^2)^2 \geq 24Rrs^2 \to (a)$$

$$\text{Also, } s^2 + 4Rr + r^2 = 18Rr + s^2 - 14Rr + r^2$$

$$= 18Rr + s^2 - 16Rr + 5r^2 + 2r(R-2r) \overset{\text{Gerretsen and Euler}}{\geq} 18Rr \Rightarrow s^2 + 4Rr + r^2 \geq 18Rr$$

$$\to (b) \therefore (a) \bullet (b) \Rightarrow ③ \text{ is true} \therefore \frac{h_a \cdot \sqrt{h_a}}{w_a} + \frac{h_b \cdot \sqrt{h_b}}{w_b} + \frac{h_c \cdot \sqrt{h_c}}{w_c} \geq \frac{3\sqrt{6}r}{\sqrt{R}} \text{ and so,}$$

$$\frac{3\sqrt{6}r}{\sqrt{R}} \leq \sum_{cyc} \frac{h_a \cdot \sqrt{h_a}}{w_a} \leq \frac{2s^2}{3\sqrt{3r}.R} \quad \forall \Delta ABC, \text{"}=\text{" iff } \Delta ABC \text{ is equilateral (QED)}$$