ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$p^2r\left(2R-r+2\sqrt{R(R-2r)}\right) \ge m_a w_a h_a r_a \ge \frac{4p^2r^3\left(2R-r-2\sqrt{R(R-2r)}\right)}{R^2}$$

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We shall first prove that : $m_a \le 2R - r + 2.\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC$

$\boxed{Case\ 1}$ $\hat{A}$ is acute and then : $m_a \le 2R\cos^2\frac{A}{2} \overset{?}{\le} 2R - r + 2.\sqrt{R(R-2r)} \Leftrightarrow$

$2Rs^2 - 2Rs(c-s) + 2R.\sqrt{1-4sc+4s^2} \overset{?}{\ge} 0 \left(c = \cos\frac{B-C}{2} \text{ and } s = \sin\frac{A}{2}\right)$

$\Leftrightarrow \sqrt{1-4sc+4s^2} \underset{①}{\boxed{\overset{?}{\ge}}} sc - 2s^2$ which is trivially true if $sc - 2s^2 < 0$ and

so, we now focus on the scenario when : $sc - 2s^2 \ge 0$ and then :

$① \Leftrightarrow 1 - 4sc + 4s^2 \underset{②}{\overset{?}{\ge}} s^2c^2 + 4s^4 - 4cs^3$ and $\because c \le 1 \therefore$ in order to prove ②,

it suffices to prove : $1 - 4sc + 4s^2 \overset{?}{\ge} s^2 + 4s^4 - 4cs^3$

$\Leftrightarrow 1 - s^2 + 4s^2(1-s^2) - 4sc(1-s^2) \overset{?}{\ge} 0 \Leftrightarrow (1-s^2)(1-4sc+4s^2) \boxed{\overset{?}{\ge}} 0$

$\Leftrightarrow \cos^2\frac{A}{2}.\frac{R-2r}{R} \overset{?}{\ge} 0 \to$ true $\Rightarrow ② \Rightarrow ①$ is true $\therefore m_a \le 2R - r + 2.\sqrt{R(R-2r)}$

$\boxed{Case\ 2}$ $\hat{A} \ge \frac{\pi}{2}$ and then : $4m_a^2 = 2b^2 + 2c^2 - 2a^2 + a^2 = 4bc\cos A + a^2 \le a^2$

$\Rightarrow m_a \le \frac{a}{2} = R\sin A \le R \overset{?}{\le} 2R - r + 2.\sqrt{R(R-2r)} \Leftrightarrow R - r + 2.\sqrt{R(R-2r)} \overset{?}{\ge} 0$

$\to$ true (strict inequality) $\therefore$ combining both cases,

$m_a \le 2R - r + 2.\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC \to$ (m)

We shall now prove that : $h_a \ge R + r - \sqrt{R^2-4r^2}\ \forall\ \Delta\ ABC$

Now, $\sqrt{R^2-4r^2} = \sqrt{(R-2r)(R+2r)} = \sqrt{R(1-4sc+4s^2)\left(R+4Rs(c-s)\right)}$

$= R.\sqrt{(1-4sc+4s^2)\left(1+4s(c-s)\right)} \ge R.\sqrt{1-4sc+4s^2}$

$\left(\because c = \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2} = s\right) \therefore h_a + \sqrt{R^2-4r^2} \ge$

$2R(c^2-s^2) + R.\sqrt{1-4sc+4s^2} \overset{?}{\ge} R + r = R + 2Rs(c-s)$

$\Leftrightarrow \sqrt{1-4sc+4s^2} \underset{③}{\boxed{\overset{?}{\ge}}} 1 + 2sc - 2c^2$ and its's trivially true when $1 + 2sc - 2c^2$

< 0 and so we now focus on the scenario when : $1 + 2sc - 2c^2 \ge 0$ and then :

$$③ \Leftrightarrow 1-4sc+4s^2 \geq (1+2sc-2c^2)^2 \Leftrightarrow -c^4+2c^3s-c^2s^2+c^2-2cs+s^2$$
$$\geq 0 \Leftrightarrow -c^2(c-s)^2+(c-s)^2 \geq 0 \Leftrightarrow (c-s)^2(1-c^2) \geq 0 \to \text{true}$$
$$\because 1 \geq \cos^2\frac{B-C}{2} \Rightarrow ③ \text{ is true} \therefore h_a \geq R+r-\sqrt{R^2-4r^2}\ \forall\ \Delta\ ABC \to (n)$$

We shall denote the semi – perimeter by "p" and via Lascu + A – G, $m_a w_a h_a r_a$

$$\geq r_b r_c r_a h_a \overset{\text{via (n)}}{\geq} rp^2\left(R+r-\sqrt{R^2-4r^2}\right) \overset{?}{\geq} \frac{4p^2r^3\left(2R-r-2.\sqrt{R(R-2r)}\right)}{R^2}$$
$$\Leftrightarrow R^3+R^2r-8Rr^2+4r^3 \overset{?}{\geq} R^2.\sqrt{R^2-4r^2}-8r^2.\sqrt{R(R-2r)}$$
$$\Leftrightarrow t^3+t^2-8t+4-t^2.\sqrt{t^2-4}+8.\sqrt{t^2-2t} \overset{?}{\geq} 0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow \left(t^3+t^2-8t+4+8.\sqrt{t^2-2t}\right)^2 \overset{?}{\geq} t^4(t^2-4)$$
$$\left(\because t^3+t^2-8t+4=(t-2)(t^2+3t-2) \overset{\text{Euler}}{\geq} 0\right)$$
$$\Leftrightarrow (t-2)(2t^4-7t^3-22t^2+92t-8)+16(t-2)(t^2+3t-2).\sqrt{t^2-2t} \overset{?}{\geq} 0$$
$$\Leftrightarrow \frac{(t-2)}{128}\left((16t^2+48t-33)(4t-13)^2+232t+4553\right)+$$
$$16(t-2)(t^2+3t-2).\sqrt{t^2-2t} \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\therefore m_a w_a h_a r_a \boxed{\geq} \frac{4p^2r^3\left(2R-r-2.\sqrt{R(R-2r)}\right)}{R^2}$$

$$\text{Again, } m_a w_a h_a r_a \overset{\text{via (m)}}{\leq} \left(2R-r+2.\sqrt{R(R-2r)}\right).\frac{2bc}{b+c}.\cos\frac{A}{2}.\frac{rp^2}{2R}.\sec^2\frac{A}{2}$$
$$\overset{?}{\leq} p^2r\left(2R-r+2.\sqrt{R(R-2r)}\right) \Leftrightarrow \frac{4R^2.\sin B\sin C}{4R^2.\cos\frac{B-C}{2}} \overset{?}{\leq} \cos^2\frac{A}{2}$$
$$\Leftrightarrow \cos(B-C)+\cos A \overset{?}{\leq} 2\cos^2\frac{A}{2}\cos\frac{B-C}{2} \Leftrightarrow c(1-s^2) \overset{?}{\geq} c^2-s^2$$
$$\Leftrightarrow c(1-c)+s^2(1-c) \overset{?}{\geq} 0 \Leftrightarrow (1-c)(c+s^2) \overset{?}{\geq} 0 \to \text{true} \because 1 \geq \cos\frac{B-C}{2}$$
$$\therefore m_a w_a h_a r_a \boxed{\leq} p^2r\left(2R-r+2.\sqrt{R(R-2r)}\right) \text{ and so,}$$
$$p^2r\left(2R-r+2.\sqrt{R(R-2r)}\right) \geq m_a w_a h_a r_a \geq \frac{4p^2r^3\left(2R-r-2.\sqrt{R(R-2r)}\right)}{R^2}$$

$\forall\ \Delta\ ABC$," = " iff Δ ABC is equilateral (QED)