ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r}\le\frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$$

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We shall first prove that : $w_a\le R+r+\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC$

$$w_a=\frac{2bc}{b+c}.\cos\frac{A}{2}=\frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$=\frac{R\left(2\cos^2\frac{B-C}{2}-1+1-2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}}=\frac{2R(c^2-s^2)}{c}\ \left(c=\cos\frac{B-C}{2}\ and\ s=\sin\frac{A}{2}\right)$$

$$\overset{?}{\le}R+r+\sqrt{R(R-2r)}$$

$$=R+2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)+R.\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 2c-\frac{2s^2}{c}\overset{?}{\le}1+2sc-2s^2+\sqrt{1-4sc+4s^2}$$

$$\Leftrightarrow 1+2sc-2c+\frac{2s^2}{c}-2s^2+\sqrt{1-4sc+4s^2}\ \underset{(1)}{\overset{?}{\ge}}\ 0$$

Now, $\frac{2s^2}{c}-2s^2=\frac{2s^2\left(1-\cos\frac{B-C}{2}\right)}{\cos\frac{B-C}{2}}\ge 0$ and $1-4sc+4s^2\overset{0<c\le 1}{\ge}1-4s+4s^2$

$=(1-2s)^2\Rightarrow\sqrt{1-4sc+4s^2}\ge|1-2s|$ and so, in order to prove (1),

it suffices to prove : $1+2sc-2c+|1-2s|\ \underset{(2)}{\ge}\ 0$

Case 1: $1-2s\ge 0$ and then : LHS of (2) $=1+2sc-2c+1-2s$

$=1-c-c(1-2s)+1-2s=(1-2s)(1-c)+(1-c)=2(1-c)(1-s)\ge 0$

$\because c=\cos\frac{B-C}{2}\le 1$ and $s=\sin\frac{A}{2}<1\Rightarrow$ (2) is true

Case 2: $1-2s<0$ and then : LHS of (2) $=1+2sc-2c+2s-1$

$=1-c+c(2s-1)+(2s-1)\overset{c\le 1}{\ge}(2s-1)(1+c)>0\ \because 1-2s<0$

$\Rightarrow$ (2) is true (strict inequality) $\therefore$ combining both cases,

(2) is true $\forall\ \Delta\ ABC\ \therefore w_a\le R+r+\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC\rightarrow$ (m)

We shall now prove that : $m_a\le 2R-r+2.\sqrt{R(R-2r)}\ \forall\ \Delta\ ABC$

Case 1: $\hat{A}$ is acute and then : $m_a\le 2R\cos^2\frac{A}{2}\overset{?}{\le}2R-r+2.\sqrt{R(R-2r)}$

$$\Leftrightarrow 2Rs^2-2Rs(c-s)+2R.\sqrt{1-4sc+4s^2}\overset{?}{\ge}0\Leftrightarrow\sqrt{1-4sc+4s^2}\ \underset{(3)}{\overset{?}{\ge}}\ sc-2s^2$$

which is trivially true if $sc-2s^2<0$ and so, we now focus on the scenario

$$\text{when}: sc-2s^2\geq 0 \text{ and then}: ③ \Leftrightarrow 1-4sc+4s^2 \underset{④}{\boxed{\overset{?}{\geq}}} s^2c^2+4s^4-4cs^3 \text{ and}$$

$$\because c\leq 1 \therefore \text{ in order to prove } ④, \text{ it suffices to prove}:$$

$$1-4sc+4s^2 \overset{?}{\geq} s^2+4s^4-4cs^3 \Leftrightarrow 1-s^2+4s^2(1-s^2)-4sc(1-s^2)\overset{?}{\geq} 0$$

$$\Leftrightarrow (1-s^2)(1-4sc+4s^2)\boxed{\overset{?}{\geq}} 0 \Leftrightarrow \cos^2\frac{A}{2}.\frac{R-2r}{R}\overset{?}{\geq} 0 \to \text{true}$$

$$\Rightarrow ④ \Rightarrow ③ \text{ is true} \therefore m_a\leq 2R-r+2.\sqrt{R(R-2r)}$$

$$\boxed{Case\ 2}\ \hat{A}\geq\frac{\pi}{2} \text{ and then}: 4m_a^2=2b^2+2c^2-2a^2+a^2=4bc\cos A+a^2\leq a^2$$

$$\Rightarrow m_a\leq\frac{a}{2}=R\sin A\leq R\overset{?}{\leq} 2R-r+2.\sqrt{R(R-2r)} \Leftrightarrow R-r+2.\sqrt{R(R-2r)}\overset{?}{\geq} 0$$

$$\to \text{true (strict inequality)} \therefore \text{combining both cases,}$$

$$m_a\leq 2R-r+2.\sqrt{R(R-2r)}\ \forall\,\Delta\,ABC \to (n)$$

$$\text{We have}: \frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r}\leq\frac{m_a}{R}+\frac{w_b}{s}+\frac{w_c}{r}\overset{\text{via (m) and (n)}}{\leq}$$

$$\frac{2R-r+2.\sqrt{R(R-2r)}}{R}+\frac{R+r+\sqrt{R(R-2r)}}{s}+\frac{R+r+\sqrt{R(R-2r)}}{r}\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq}$$

$$\frac{2R-r+2.\sqrt{R(R-2r)}+2R+2r+2.\sqrt{R(R-2r)}}{2r}+\frac{\sqrt{3}\left(R+r+\sqrt{R(R-2r)}\right)}{9r}$$

$$=\frac{(36+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)+(9+2\sqrt{3})r}{18r}\overset{?}{\leq}\frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$$

$$\Leftrightarrow (81+6\sqrt{3}-72-4\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)\overset{?}{\geq} 2r(9+2\sqrt{3})$$

$$\Leftrightarrow (9+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)\overset{?}{\geq} 2r(9+2\sqrt{3}) \Leftrightarrow R-2r+\sqrt{R(R-2r)}\overset{?}{\geq} 0$$

$$\to \text{true} \because R\overset{\text{Euler}}{\geq} 2r \therefore \frac{m_a}{R}+\frac{w_b}{s}+\frac{h_c}{r}\leq\frac{(27+2\sqrt{3})\left(R+\sqrt{R(R-2r)}\right)}{12r}$$

$$\forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$