ROMANIAN MATHEMATICAL MAGAZINE

In the acute $\Delta ABC, k=\frac{\sqrt[4]{3}(673\sqrt{3}+441)}{598}$. Prove that :

$$\frac{1}{s-R}+\frac{1}{R-r}+\frac{1}{s-r}\le\frac{k}{\sqrt{F}}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$s-r=\sqrt{s^2}-\sqrt{r^2}\overset{Mitrinovic}{\ge}\sqrt{3\sqrt{3}sr}-\sqrt{\frac{sr}{3\sqrt{3}}}=(3\sqrt{3}-1)\sqrt{\frac{F}{3\sqrt{3}}}$$

$$R-r\overset{Mitrinovic}{\ge}\frac{2s}{3\sqrt{3}}-r=2\sqrt{\frac{s^2}{27}}-\sqrt{r^2}\overset{Mitrinovic}{\ge}2\sqrt{\frac{sr}{3\sqrt{3}}}-\sqrt{\frac{sr}{3\sqrt{3}}}=\sqrt{\frac{F}{3\sqrt{3}}}$$

$$s-R=\sqrt{s}\left(\sqrt{s}-\frac{R}{\sqrt{s}}\right)\overset{Walker}{\ge}\sqrt{s}\left(\sqrt[4]{2R^2+8Rr+3r^2}-\frac{R}{\sqrt[4]{2R^2+8Rr+3r^2}}\right)=\sqrt{F}.f\left(\frac{R}{r}\right)$$

where $f(x)=\sqrt[4]{2x^2+8x+3}-\frac{x}{\sqrt[4]{2x^2+8x+3}}, x\ge 0.$ It is easy to find that

$$f'(x)=\frac{(x+2)\sqrt{2x^2+8x+3}-(x^2+6x+3)}{\sqrt[4]{2x^2+8x+3}}$$
$$=\frac{x^4+4x^3+x^2+8x+3}{\sqrt[4]{2x^2+8x+3}\left((x+2)\sqrt{2x^2+8x+3}+x^2+6x+3\right)}>0,$$

$$\Rightarrow s-R\ge f\left(\frac{R}{r}\right).\sqrt{F}\overset{Euler}{\ge}f(2).\sqrt{F}=(3\sqrt{3}-2)\sqrt{\frac{F}{3\sqrt{3}}}.$$

Therefore

$$\frac{1}{s-R}+\frac{1}{R-r}+\frac{1}{s-r}\le\frac{\sqrt{3\sqrt{3}}}{(3\sqrt{3}-2)\sqrt{F}}+\sqrt{\frac{3\sqrt{3}}{F}}+\frac{\sqrt{3\sqrt{3}}}{(3\sqrt{3}-1)\sqrt{F}}=\frac{k}{\sqrt{F}}.$$

Equality holds iff ΔABC is equilateral.