ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$4\sqrt{2Rsr}\prod_{cyc}\sqrt{m_a-h_a}\geq|(a-b)(b-c)(c-a)|\geq 4R\sqrt{r}\prod_{cyc}\sqrt{w_a-h_a}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\rightarrow(1)$$

$$\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\rightarrow(2)$$

$$\text{and},\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\rightarrow(3)$$

$$\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{1}{64}\left(\begin{matrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2-576Rrs^2(s^2-4Rr-r^2)\\-432R^2r^2s^2\end{matrix}}{64}$$

$$\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3}{16}\rightarrow(m)$$

$$\text{Now,}\left(4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\right)^2=32Rsr.\prod_{cyc}\frac{m_a^2-h_a^2}{m_a+h_a}\geq$$

$$\frac{4Rsr}{m_am_bm_c}.\prod_{cyc}\left(\frac{(b-c)^2}{4}+\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$=\frac{4Rsr}{m_am_bm_c}.\frac{1}{64.16R^2r^2s^2}.\prod_{cyc}(b+c)^2.\prod_{cyc}(b-c)^2\overset{?}{\geq}\prod_{cyc}(b-c)^2$$

$$\Leftrightarrow s(s^2+2Rr+r^2)^2\overset{?}{\geq}64Rr\prod_{cyc}m_a\overset{\text{via (m)}}{\Leftrightarrow}s^2(s^2+2Rr+r^2)^4\geq$$

$$256R^2r^2(s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3)$$

$$\Leftrightarrow s^2(s^2+2Rr+r^2)^4$$

$$-256R^2r^2(s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3)$$

$$\underset{(*)}{\overset{?}{\geq}}0\ \text{and}\ \because P=(s^2-16Rr+5r^2)^5+(88Rr-21r^2)(s^2-16Rr+5r^2)^4$$

$$+8r^2(355R^2-185Rr+22r^2)(s^2-16Rr+5r^2)^3$$

$$+8r^3(5652R^3-5463R^2r+1164Rr^2-92r^3)(s^2-16Rr+5r^2)^2$$

$$+16r^4(24705R^4-37890R^3r+14868R^2r^2-1624Rr^3+96r^4)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove }(*),\text{it suffices to prove : LHS of }(*)\geq P$$

$$\Leftrightarrow 104976t^5-203877t^4+129384t^3-28152t^2+1696t-80\geq 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$

$$(t-2)(104976t^4+6075t^3+141534t^2+254916t+511528)+1022976\geq 0$$

$$\rightarrow \text{true (strict inequality)}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore 4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq|(a-b)(b-c)(c-a)|$$

$$\text{Again,}\left(4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}\right)^2=16R^2r.\prod_{cyc}\frac{w_a^2-h_a^2}{w_a+h_a}\leq$$

$$\frac{2R^2r}{h_ah_bh_c}.\prod_{cyc}\left(\frac{s(s-a)(b-c)^2}{a^2}-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$=\frac{2R^3r}{2r^2s^2}.\frac{\prod_{cyc}(4s^2(s-a)^2)}{16R^2r^2s^2.\prod_{cyc}(b+c)^2}.\prod_{cyc}(b-c)^2$$

$$=\frac{R^3}{rs^2}.\frac{64s^6.r^4s^2}{16R^2r^2s^2.4s^2(s^2+2Rr+r^2)^2}.\prod_{cyc}(b-c)^2\overset{?}{\leq}\prod_{cyc}(b-c)^2$$

$$\Leftrightarrow(s^2+2Rr+r^2)^2\overset{?}{\geq}Rrs^2\rightarrow\text{true (strict inequality)}\because s^2+2Rr+r^2>s^2,Rr$$

$$\therefore 4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}\leq|(a-b)(b-c)(c-a)|\ \text{and so,}$$

$$4.\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq|(a-b)(b-c)(c-a)|\geq 4R.\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a}$$

$$\forall\, \Delta\, \text{ABC}, ''='' \text{ iff } \Delta\, \text{ABC is isosceles (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that

$$2a(m_a - h_a) \geq (b-c)^2 \geq \frac{a^2(w_a - h_a)}{\sqrt{s(s-a)}}. \quad (1)$$

We have

$$2a(m_a - h_a) \overset{?}{\geq} (b-c)^2 \Leftrightarrow 2am_a \geq 4F + (b-c)^2$$

$$\Leftrightarrow a^2(2b^2 + 2c^2 - a^2) \geq 16F^2 + 8F(b-c)^2 + (b-c)^4$$

$$\Leftrightarrow a^2(2b^2 + 2c^2 - a^2) - 2(a^2b^2 + b^2c^2 + c^2a^2) + (a^4 + b^4 + c^4) \geq$$

$$\geq 8F(b-c)^2 + (b-c)^4$$

$$\Leftrightarrow (b^2 - c^2)^2 \geq 8F(b-c)^2 + (b-c)^4 \Leftrightarrow 4(b-c)^2(bc - 2F) \geq 0$$
$$\Leftrightarrow 4(b-c)^2 bc(1 - \sin A) \geq 0,$$

which is true. So the proof of the left side of inequality (1) is complete.

$$(b-c)^2 \overset{?}{\geq} \frac{a^2(w_a - h_a)}{\sqrt{s(s-a)}} \Leftrightarrow (b-c)^2 \geq a^2\left(\frac{2\sqrt{bc}}{b+c} - \frac{2\sqrt{(s-b)(s-c)}}{a}\right)$$

$$\Leftrightarrow (b-c)^2 \geq a.\frac{4a^2bc - (b+c)^2[a^2 - (b-c)^2]}{2(b+c)\left(a\sqrt{bc} + (b+c)\sqrt{(s-b)(s-c)}\right)}$$

$$\Leftrightarrow 1 \geq \frac{a.4s(s-a)}{2(b+c)\left(a\sqrt{bc} + (b+c)\sqrt{(s-b)(s-c)}\right)},$$

which is true because $b + c = s + (s-a) \geq 2\sqrt{s(s-a)}$,

$$bc = s(s-a) + (s-b)(s-c) \geq s(s-a).$$

So the proof of the right side of inequality (1) is complete.

Using the inequality (1), we have

$$\prod_{cyc}\sqrt{2a}.\sqrt{m_a - h_a} \geq |(a-b)(b-c)(c-a)| \geq \prod_{cyc}\frac{a}{\sqrt[4]{s(s-a)}}.\sqrt{w_a - h_a}$$

$$\Leftrightarrow 4\sqrt{2Rsr}.\prod_{cyc}\sqrt{m_a-h_a}\geq|(a-b)(b-c)(c-a)|\geq 4R\sqrt{r}.\prod_{cyc}\sqrt{w_a-h_a},$$

as desired. Equality holds iff ΔABC is isosceles.