ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$3-\sqrt{2}+\frac{R}{\sqrt{2}r}\ge\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\ge 3+\frac{3(R-2r)}{2(13R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{m_a}{w_a}=\sum_{cyc}\frac{(b+c)m_a}{2\sqrt{bc.s(s-a)}}\overset{CBS}{\le}\frac{1}{2}\sqrt{\sum_{cyc}\frac{(b+c)^2}{bc}\cdot\sum_{cyc}\frac{{m_a}^2}{s(s-a)}},$$

with

$$\sum_{cyc}\frac{(b+c)^2}{bc}=\frac{\sum_{cyc}a(b+c)^2}{abc}=\frac{2s(s^2+r^2+10Rr)}{4Rrs}\overset{Gerretsen}{\le}\frac{4R^2+14Rr+4r^2}{2Rr}$$

$$\sum_{cyc}\frac{{m_a}^2}{s(s-a)}=\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4s(s-a)}=\frac{1}{4s}\sum_{cyc}\left(\frac{2(a^2+b^2+c^2)-3s^2}{s-a}+3(s+a)\right)=$$

$$=\frac{1}{4s}\left(\frac{[2(a^2+b^2+c^2)-3s^2](4R+r)}{sr}+15s\right)=\frac{1}{4s}\left(\frac{(s^2-4r^2-16Rr)(4R+r)}{sr}+15s\right)=$$

$$=\frac{R+4r}{r}-\frac{(4R+r)^2}{s^2}\overset{Gerretsen-Blundon}{\le}\frac{R+4r}{r}-\frac{2(2R-r)}{R}=\frac{R^2+2r^2}{Rr}$$

then

$$\sum_{cyc}\frac{m_a}{w_a}\le\frac{\sqrt{(R^2+2r^2)(2R^2+7Rr+2r^2)}}{2Rr}\overset{?}{\le}3-\sqrt{2}+\frac{R}{\sqrt{2}r}$$

$$\overset{squaring}{\Leftrightarrow}\frac{(R-2r)\left(3(4\sqrt{2}-5)R^2+8Rr+2r^2\right)}{4R^2r}\ge 0$$

which is true and the proof of the left side of the desired inequality is complete.

$$\sum_{cyc}\frac{m_a}{w_a}=\sum_{cyc}\frac{m_aw_a}{{w_a}^2}\ge\sum_{cyc}\frac{s(s-a)}{{w_a}^2}=\sum_{cyc}\frac{(b+c)^2}{4bc}=\frac{\sum_{cyc}a(b+c)^2}{4abc}=\frac{2s(s^2+r^2+10Rr)}{16Rrs}$$

$$\overset{Gerretsen}{\ge}\frac{26Rr-4r^2}{8Rr}=3+\frac{R-2r}{4R}\overset{Euler}{\ge}3+\frac{3(R-2r)}{2(13R-2r)}$$

So the proof is complete. Equality holds iff ΔABC is equilateral.