

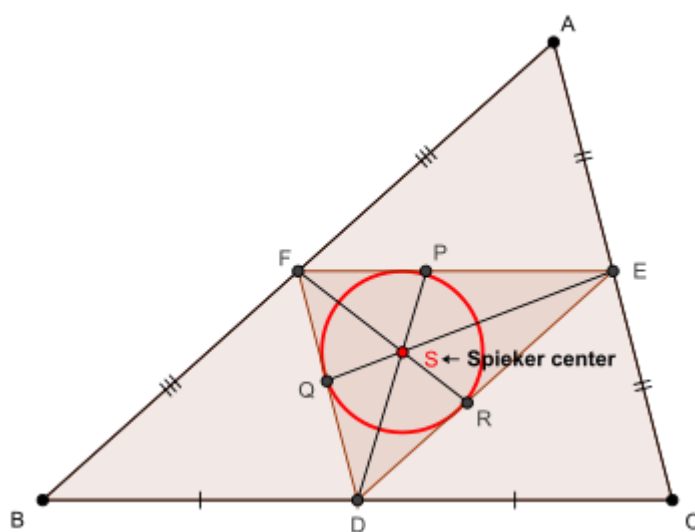
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In any $\triangle ABC$, the following relationship holds :

$$\frac{h_a}{m_a} + \frac{w_a}{g_a} + \frac{p_a}{n_a} \leq \frac{g_a}{h_a} + \frac{m_a}{p_a} + \frac{n_a}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\sphericalangle BAX) = \alpha$ and $m(\sphericalangle CAX) = \beta$ (say)
and inradius of $\triangle DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \triangle DEF, \therefore m(\sphericalangle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\sphericalangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on $\triangle AFS$ and $\triangle AES$, we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2} \\ \Rightarrow 2AS^2 &\stackrel{(i)}{=} \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 \text{Now, } & \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & = \frac{r}{2} \left(4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right) \\
 & = 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 & = \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 & = \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 & = \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 & \Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & \quad \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 \text{Again, } & \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 & = \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 & \text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 & = \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s} \\
 & = \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 \text{Via sine law on } \triangle AFS, & \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}} \\
 \Rightarrow c\sin\alpha & \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS} \\
 \text{Now, [BAX] + [BAX]} & = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 \text{via (***) and (***)} & \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS
 \end{aligned}$$

$$\begin{aligned} \Rightarrow p_a^2 &\stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s} \\ \therefore p_a^2 &\stackrel{(\circ)}{=} \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2)) \\ \text{Now, } b^3 + c^3 - abc + a(4m_a^2) &= b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2) \\ &= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2) \\ &= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2) \\ &= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right) \\ &= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \cdot \frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4} \\ &\quad - \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y) \\ &= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2} \\ \therefore b^3 + c^3 - abc + a(4m_a^2) &\stackrel{(\bullet\bullet)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2} \\ \therefore (\circ), (\bullet\bullet) \Rightarrow p_a^2 &= \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right) \\ &= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right) \\ &= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2 \\ &= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right) \\ \Rightarrow p_a^2 &\stackrel{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \\ \text{Now, } m_a n_a &\stackrel{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \stackrel{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \end{aligned}$$

$$\begin{aligned}
 & \left(s(s-a) + \frac{(b-c)^2}{4} \right) \left(s(s-a) + \frac{s(b-c)^2}{a} \right) \stackrel{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right)^2 \\
 & \quad + \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} \cdot \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right) \\
 & \Leftrightarrow s(s-a)(b-c)^2 \left(\frac{s}{a} + \frac{1}{4} \right) + \frac{s(b-c)^4}{4a} \stackrel{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} + \\
 & 2s(s-a) \cdot \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a) \cdot \frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2} \\
 & \Leftrightarrow s(s-a) \left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9} \right) + \\
 & \left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2} \right) (b-c)^2 \stackrel{?}{\geq} 0 \quad (\because (b-c)^2 \geq 0) \\
 & \Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} \\
 & + \frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4} \cdot (b-c)^2 \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow \frac{s(s-a) \left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3 \right)}{36a(2s+a)^2} + \\
 & \frac{(s-a) \left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4 \right)}{324a(2s+a)^4} \cdot (b-c)^2 \\
 & \stackrel{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow \frac{m_a}{p_a} \geq \frac{p_a}{n_a} \rightarrow \textcircled{3}
 \end{aligned}$$

Finally, $an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s-a)^2 \Leftrightarrow$

$$\left(b^2(s-c) + c^2(s-b) - a(s-b)(s-c) \right) \left(b^2(s-b) + c^2(s-c) - a(s-b)(s-c) \right) \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2$$

Let $s-a = x, s-b = y$ and $s-c = z \therefore s = x+y+z \Rightarrow a = y+z,$

$b = z+x$ and $c = x+y$

Via such substitutions, (a) \Leftrightarrow

$$\begin{aligned}
 & \left(z(z+x)^2 + y(x+y)^2 - yz(y+z) \right) \left(y(z+x)^2 + z(x+y)^2 - yz(y+z) \right) \\
 & \geq x^2(y+z)^2(x+y+z)^2 \\
 & \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \\
 & \rightarrow \text{true} \Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \geq s(s-a) \geq w_a^2 \Rightarrow \frac{n_a}{w_a} \geq \frac{w_a}{g_a} \rightarrow \textcircled{4}
 \end{aligned}$$

Also, $m_a g_a \geq h_a^2 \Rightarrow \frac{g_a}{h_a} \geq \frac{h_a}{m_a} \rightarrow \textcircled{5} \therefore \textcircled{3} + \textcircled{4} + \textcircled{5} \Rightarrow \frac{h_a}{m_a} + \frac{w_a}{g_a} + \frac{p_a}{n_a} \leq$

$$\frac{g_a}{h_a} + \frac{m_a}{p_a} + \frac{n_a}{w_a} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$