ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$4-\frac{2r}{R}\le\frac{m_ah_a}{w_ar_a}+\frac{m_bh_b}{w_br_b}+\frac{m_ch_c}{w_cr_c}\le\frac{2\sqrt{2}R^2+2(3-4\sqrt{2})r^2}{Rr}$$

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$$\frac{m_ah_a}{w_ar_a}+\frac{m_bh_b}{w_br_b}+\frac{m_ch_c}{w_cr_c}=\sum_{cyc}\frac{m_abc(b+c)}{2R.2bc\cos\frac{A}{2}.s\tan\frac{A}{2}}$$

$$=\sum_{cyc}\frac{m_a(b+c).\sqrt{bc}.\sqrt{s-a}}{4Rs.\sqrt{(s-b)(s-c)(s-a)}}=\frac{1}{4Rrs.\sqrt{s}}.\sum_{cyc}\left(m_a.\sqrt{bc}.(b+c).\sqrt{s-a}\right)$$

$$\overset{CBS}{\le}\frac{1}{4Rrs.\sqrt{s}}.\sqrt{\sum_{cyc}(bc.m_a^2)}.\sqrt{\sum_{cyc}\left((s-a)(b+c)^2\right)}$$

$$=\frac{1}{4Rrs.\sqrt{s}}.\sqrt{\frac{1}{4}\sum_{cyc}\left(bc\left(2\sum_{cyc}a^2-3a^2\right)\right)}.\sqrt{\sum_{cyc}\left((s-a)(4s^2-4sa+a^2)\right)}$$

$$=\frac{1}{4Rrs.\sqrt{s}}\sqrt{\frac{1}{4}\left(\begin{matrix}4(s^2-4Rr-r^2)(s^2+4Rr+r^2)\\-24Rrs^2\end{matrix}\right)}\sqrt{\begin{matrix}4s(s^2-2(4Rr+r^2))+\\2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\end{matrix}}$$

$$=\frac{1}{2Rrs}.\sqrt{s^4-6Rrs^2-r^2(4R+r)^2}.\sqrt{s^2-7Rr-r^2}\overset{?}{\le}\frac{2\sqrt{2}R^2+2(3-4\sqrt{2})r^2}{Rr}$$

$$\Leftrightarrow(s^4-6Rrs^2-r^2(4R+r)^2)(s^2-7Rr-r^2)$$

$$\boxed{\overset{?}{\le}}\ \text{①}\ 4s^2\left(8(R^2-4r^2)^2+36r^4+24\sqrt{2}r^2(R^2-4r^2)\right)$$

Now, $R^2-4r^2\overset{Euler}{\ge}0$ and $24\sqrt{2}>33$ $\therefore$ RHS of ① $\ge$

$$4s^2\left(8(R^2-4r^2)^2+36r^4+33r^2(R^2-4r^2)\right)$$

$$\overset{?}{\ge}(s^4-6Rrs^2-r^2(4R+r)^2)(s^2-7Rr-r^2)$$

$$\Leftrightarrow s^6-(13Rr+r^2)s^4-(32R^4-150R^2r^2+2Rr^3+129r^4)s^2$$

$$+r^3(112R^3+72R^2r+15Rr^2+r^3)\ \boxed{\overset{?}{\le}}\ \text{②}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge0$ and $s^2-(m+n)\le0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \overset{(*)}{\le} 0$$

$$\Rightarrow \mathbf{P} = s^2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) +$$

$$(4R^2 + 7Rr - 3r^2)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \overset{via\ (*)}{\le} 0$$

∴ in order to prove ②, it suffices to prove : LHS of ② ≤ P

$$\Leftrightarrow (8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4)s^2$$

$$+r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) \overset{③}{\ge} 0$$

Case 1 $8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4 \ge 0$ *and then* : LHS of ③ ≥ $r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) > 0\ (\because R \ge 2r)$

Case 2 $8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4 < 0$ *and then* : LHS of ③ $\overset{Gerretsen}{\ge} (8R^4 - 22R^3r - 111R^2r^2 + 44Rr^3 + 62r^4)(4R^2 + 4Rr + 3r^2)$

$$+r(128R^5 + 320R^4r + 40R^3r^2 - 64R^2r^3 - 22Rr^4 - 2r^5) \overset{?}{\ge} 0$$

$$\Leftrightarrow 32t^6 + 72t^5 - 188t^4 - 294t^3 + 27t^2 + 358t + 184 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(32t^4 + 200t^3 + 484t^2 + 842t + 1459) + 2826\right) \overset{?}{\ge} 0 \to \text{true}$$

$\because t \overset{Euler}{\ge} 2 \Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\therefore \frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \le \frac{2\sqrt{2}R^2 + 2(3 - 4\sqrt{2})r^2}{Rr}$$

Again, $\frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \ge \sum_{cyc} \frac{h_a}{r_a} = \sum_{cyc} \frac{2rs}{4Rs\cos^2\frac{A}{2}\tan^2\frac{A}{2}} = \frac{r}{2R} \cdot \sum_{cyc} \frac{bc(s-a)}{r^2 s}$

$$= \frac{s(s^2 - 8Rr + r^2)}{2Rrs} \overset{Gerretsen}{\ge} \frac{8R - 4r}{2R} = 4 - \frac{2r}{R} \text{ and so,}$$

$$4 - \frac{2r}{R} \le \frac{m_a h_a}{w_a r_a} + \frac{m_b h_b}{w_b r_b} + \frac{m_c h_c}{w_c r_c} \le \frac{2\sqrt{2}R^2 + 2(3 - 4\sqrt{2})r^2}{Rr}\ \forall \Delta ABC,$$

'' ='' iff Δ ABC is equil*ateral* (QED)