In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{h_a\sqrt{h_a}}+\frac{m_b^2}{h_b\sqrt{h_b}}+\frac{m_c^2}{h_c\sqrt{h_c}}\le\frac{R^2+Rr+(3\sqrt{6}-6)r^2}{r\sqrt{2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2}{h_a.\sqrt{h_a}}+\frac{m_b^2}{h_b.\sqrt{h_b}}+\frac{m_c^2}{h_c.\sqrt{h_c}}\le\frac{R^2+Rr+(3\sqrt{6}-6)r^2}{r.\sqrt{2r}}$$

$$\Leftrightarrow\sum_{cyc}\frac{a.\sqrt{a}.m_a^2}{2rs.\sqrt{2rs}}\le\frac{R^2+Rr+(3\sqrt{6}-6)r^2}{r.\sqrt{2r}}$$

$$\Leftrightarrow\sum_{cyc}\frac{a.\sqrt{a}.m_a^2}{2s.\sqrt{s}}\le R^2+Rr+(3\sqrt{6}-6)r^2\to(m)$$

$$\text{Now, }\sum_{cyc}\frac{a.\sqrt{a}.m_a^2}{2s.\sqrt{s}}\overset{CBS}{\le}\frac{1}{2s.\sqrt{s}}.\sqrt{\sum_{cyc}a^2m_a^2}.\sqrt{\sum_{cyc}am_a^2}$$

$$=\frac{1}{4s.\sqrt{s}}.\sqrt{\sum_{cyc}a^2(2b^2+2c^2-a^2)}.\sqrt{s\sum_{cyc}a(s-a)+\frac{1}{4}\sum_{cyc}a(b^2+c^2-2bc)}$$

$$=\frac{1}{4s.\sqrt{s}}.\sqrt{4\sum_{cyc}a^2b^2+16r^2s^2-2\sum_{cyc}a^2b^2}.\sqrt{\begin{matrix}s(2s^2-2(s^2-4Rr-r^2))\\+\dfrac{2s(s^2+4Rr+r^2)-36Rrs}{4}\end{matrix}}$$

$$=\frac{1}{4s}.\sqrt{(s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2}.\sqrt{s^2+2Rr+5r^2}$$

$$\overset{?}{\le}R^2+Rr+(3\sqrt{6}-6)r^2$$

$$\Leftrightarrow\frac{((s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2)(s^2+2Rr+5r^2)}{16s^2}$$

$$\underset{(1)}{\overset{?}{\le}}(R^2+Rr-6r^2)^2+54r^2+6\sqrt{6}r^2(R^2+Rr-6r^2)$$

$$\text{Now, }R^2+Rr-6r^2=(R-2r)(R+3r)\overset{Euler}{\ge}0\text{ and }6\sqrt{6}>14\therefore\text{RHS of (1)}\ge$$

$$(R^2+Rr-6r^2)^2+54r^2+14r^2(R^2+Rr-6r^2)$$

$$\overset{?}{\ge}\frac{((s^2+4Rr+r^2)^2-16Rrs^2+8r^2s^2)(s^2+2Rr+5r^2)}{16s^2}$$

$$\Leftrightarrow s^6-(6Rr-15r^2)s^4-r(32R^3+48R^2r+44Rr^2+45r^3)s^2$$

$$+r^3(32R^3+96R^2r+42Rr^2+5r^3)\underset{(2)}{\overset{?}{\le}}0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\ge0\text{ and }s^2-(m+n)\le0\text{, where }m=$$

$$2R^2+10Rr-r^2\text{ and }n=2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0 \Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(*)}{\le} 0$$

$$\Rightarrow \mathbf{P} = s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$
$$(4R^2+14Rr+13r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$$
$$4r(10R^3+57R^2r+44Rr^2-18r^3)(s^2-4R^2-4Rr-3r^2) \overset{\textit{via } (*) \text{ and Gerretsen}}{\le} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\le$ P

$$\Leftrightarrow 12t^5+2t^4-27t^3-50t^2-14t+28 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(12t^4+26t^3+25t^2-14) \overset{?}{\ge} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow ② \Rightarrow ①$$

is true $\therefore \dfrac{m_a^2}{h_a.\sqrt{h_a}}+\dfrac{m_b^2}{h_b.\sqrt{h_b}}+\dfrac{m_c^2}{h_c.\sqrt{h_c}} \le \dfrac{R^2+Rr+\left(3\sqrt{6}-6\right)r^2}{r.\sqrt{2r}}\ \forall\, \Delta\, ABC,$

$''=''$ iff Δ ABC is equilateral (QED)