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In any Δ ABC the following relationship holds :

$$\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\le 1+\frac{R}{r}$$

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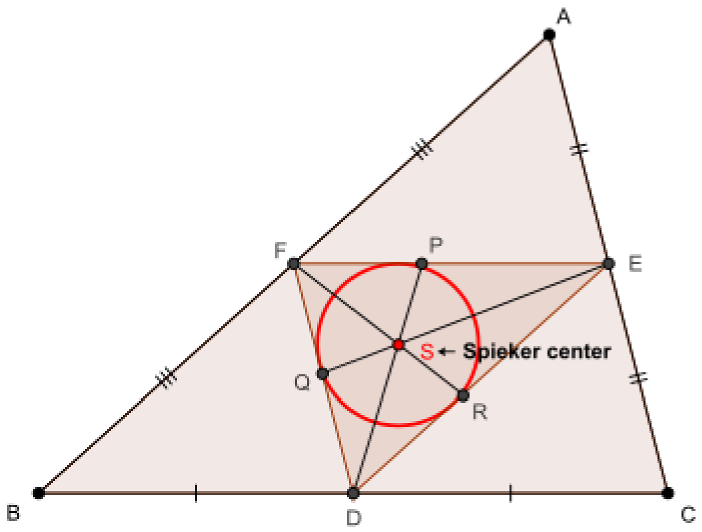

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$(say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\ \frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\ \boxed{\overset{(**)}{=}}\ \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\ \boxed{\overset{(ii)}{=}}\ \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

Now, $b^3+c^3-abc+a(4m_a^2)=b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$

$$=\sum_{cyc}a^3-abc+2a.2bc\cos A=2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$=2s\left(s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)\right)\therefore(\bullet),(\bullet\bullet)\Rightarrow$$

$$p_a\overset{(\bullet\bullet\bullet)}{=}\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}$$

We have : $\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}$

$$=\sum_{cyc}\left(\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\cdot\frac{a(b+c).\sqrt{bc}}{2abc.\sqrt{s(s-a)}}\right)$$

$$=\frac{2s}{2s(9s^2+6Rr+r^2).8Rrs}\cdot\sum_{cyc}\left(\frac{(2s+b)(2s+c)a(b+c).\sqrt{bc(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\right)$$

$$=\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sum_{cyc}\left(\frac{\sqrt{(2s+b)(2s+c)a(b+c)bc(s-b)(s-c)}.}{\sqrt{(2s+b)(2s+c)a(b+c)\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)}}\right)$$

$$\overset{CBS}{\le}\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sqrt{x}.\sqrt{y}$$

$$\left(\begin{matrix}\text{where } x=4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)\text{ and}\\ y=\sum_{cyc}\left((2s+b)(2s+c)a(b+c)\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)\right)\end{matrix}\right)$$

$$\therefore\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\overset{(1)}{\le}\frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}\cdot\sqrt{x}.\sqrt{y}$$

Now, $\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)=\sum_{cyc}\left((8s^2-2sa+bc)(s-b)(s-c)\right)$

$$= r^2 s.\sum_{cyc}\left(\frac{2s(s-a)+6s^2+bc}{s-a}\right) = r^2 s\left(6s+\frac{6s^2(4Rr+r^2)}{r^2 s}+s.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\Rightarrow \sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right) \overset{(\blacksquare)}{=} 6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2$$

and also,

$$\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right) = r^2 s.\sum_{cyc}\frac{a(2s(s-a)+6s^2+bc)}{s-a}$$

$$= r^2 s.\left(2s(2s)+6s^2.\sum_{cyc}\frac{a-s+s}{s-a}+\frac{4Rrs(4Rr+r^2)}{r^2 s}\right)$$

$$= r^2 s.\left(4s^2+6s^2.\left(-3+\frac{s(4Rr+r^2)}{r^2 s}\right)+4R(4R+r)\right)$$

$$\Rightarrow \sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right) \overset{(\blacksquare\blacksquare)}{=} r^2 s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)$$

and moreover, $4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$

$$= 4Rrs.r^2 s.\sum_{cyc}\frac{(2s+b)(2s+c)(s+s-a)}{s-a}$$

$$= \frac{4Rrs.r^2s^2}{r^2 s}.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)+$$

$$4Rrs.r^2 s.\sum_{cyc}(8s^2-2sa+bc) \overset{\text{via }(\blacksquare)}{=} 4Rrs^2(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2)$$

$$+4Rr^3s^2(21s^2+4Rr+r^2) \Rightarrow 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$= x \overset{(\blacksquare\blacksquare\blacksquare)}{=} 8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right)$$

$$\text{Now, } y = \sum_{cyc}\left((s^2-3r^2)(2s+b)(2s+c)a(b+c)\right)$$

$$-\frac{16Rr}{4Rrs}.\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\left(\sum_{cyc}ab-bc\right)\right)$$

$$= (s^2-3r^2)\sum_{cyc}\left(\left(\sum_{cyc}ab-bc\right)(8s^2-2sa+bc)\right)-$$

$$\frac{4}{s}\begin{pmatrix}(s^2+4Rr+r^2).\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)\\ -4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)\end{pmatrix}$$

$$\overset{\text{via }(\blacksquare)\text{ and }(\blacksquare\blacksquare)}{=}(s^2-3r^2)(s^2+4Rr+r^2)(21s^2+4Rr+r^2)-$$
$$(s^2-3r^2)(8s^2(s^2+4Rr+r^2)-24Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-\frac{4}{s}\begin{pmatrix}(s^2+4Rr+r^2).r^2s\left(-8s^2+\dfrac{24Rs^2}{r}+16R^2+4Rr\right)\\ -4Rrs.(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2)\end{pmatrix}$$

$$\Rightarrow y\ \boxed{\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}}\ 4s^2\left(3s^4-(2Rr-2r^2)s^2-r^2(16R^2+10Rr+r^2)\right)$$

$$\therefore ①,(\blacksquare\blacksquare\blacksquare),(\blacksquare\blacksquare\blacksquare\blacksquare)\Rightarrow\left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2\le$$

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\begin{pmatrix}3s^4-(2Rr-2r^2)s^2\\ -r^2(16R^2+10Rr+r^2)\end{pmatrix}}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4}\overset{?}{\le}\frac{(R+r)^2}{r^2}$$

$$\Leftrightarrow -(36R+51r)s^6+(162R^3+324R^2r+154Rr^2-37r^3)s^4$$
$$+r(216R^4+676R^3r+676R^2r^2+208Rr^3+15r^4)s^2$$
$$+r^2(72R^5+296R^4r+298R^3r^2+112R^2r^3+18Rr^4+r^5)\ \boxed{\overset{?}{\underset{②}{\ge}}}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ *and* $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(a)}{\le}0$$
$\Rightarrow -(36R+51r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\ge 0$ *and so, in order to prove* ②, it suffices to prove : LHS of ② $\ge$
$$-(36R+51r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$\Leftrightarrow(18R^3-600R^2r-794Rr^2+65r^3)s^4$$
$$+r(2520R^4+5668R^3r+3556R^2r^2+856Rr^3+66r^4)s^2$$
$$+r^2(72R^5+296R^4r+298R^3r^2+112R^2r^3+18Rr^4+r^5)\ \boxed{\overset{③}{\ge}}\ 0$$

We note that ③ is trivially true if : $18R^3-600R^2r-794Rr^2+65r^3\ge 0$ *and so we now focus on the case when* : $18R^3-600R^2r-794Rr^2+65r^3<0$

and then : $(18R^3-600R^2r-794Rr^2+65r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\ +r(4R+r)^3\end{pmatrix}$

$\overset{\text{via }(a)}{\ge}$ 0 *and so, in order to prove* ③, it suffices to prove :

$$\text{LHS of ③}\ge(18R^3-600R^2r-794Rr^2+65r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\ +r(4R+r)^3\end{pmatrix}$$

$$\Leftrightarrow(9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5)s^2\ \boxed{\overset{④}{\ge}}$$

$$r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

Case 1 $9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5\geq 0$ *and then* :

$$\text{LHS of } ④ \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} 9R^5+60R^4r-1193R^3r^2 \\ -1358R^2r^3+468Rr^4-8r^5 \end{pmatrix}(16Rr-5r^2)$$

$$\overset{?}{\geq} r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

$$\Leftrightarrow 2808t^5-4713t^4-5292t^3+7584t^2-1232t+16\overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2808t^4+903t^3-3486t^2+612t-8)\overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow$ ④ is true

Case 2 $9R^5+60R^4r-1193R^3r^2-1358R^2r^3+468Rr^4-8r^5<0$ *and then* :

$$\text{LHS of } ④ \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} 9R^5+60R^4r-1193R^3r^2 \\ -1358R^2r^3+468Rr^4-8r^5 \end{pmatrix}(4R^2+4Rr+3r^2)$$

$$\overset{?}{\geq} r(144R^6-4701R^5r-9962R^4r^2-5179R^3r^3-890R^2r^4-4Rr^5+8r^6)$$

$$\Leftrightarrow 18t^7+66t^6+98t^5-31t^4-980t^3-672t^2+688t-16\overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(18t^6+102t^5+302t^4+573t^3+166t^2-340t+8)\overset{?}{\geq} 0 \rightarrow \text{true}$$

$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ④ is true $\therefore$ combining both cases, ④ $\Rightarrow$ ③ $\Rightarrow$ ② is true $\forall\, \Delta\, ABC$

$$\therefore \frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\leq 1+\frac{R}{r}\ \forall\, \Delta\, ABC, \text{"="} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$