ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}\le\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} w_a=\sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right)\overset{CBS}{\le}2\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}\overset{A-G}{\le}$$

$$\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}}=\sqrt{\sum_{cyc}bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}$$

$$=\sqrt{\sum_{cyc}bc}.\sqrt{\frac{4R+r}{2R}}\Rightarrow\sum_{cyc}w_a\le\sqrt{\sum_{cyc}ab}.\sqrt{\frac{4R+r}{2R}}\to(a)$$

$$\text{Now, }\frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}=\frac{1}{\sqrt{w_a w_b w_c}}.\sum_{cyc}\sqrt{w_b w_c}\overset{CBS}{\le}$$

$$\frac{\sqrt{s^2+2Rr+r^2}}{\sqrt{16Rr^2s^2}}.\sqrt{\sum_{cyc}w_b}\sqrt{\sum_{cyc}w_c}\overset{via\,(a)}{\le}\frac{\sqrt{s^2+2Rr+r^2}}{\sqrt{16Rr^2s^2}}.\sqrt{\sum_{cyc}ab}.\sqrt{\frac{4R+r}{2R}}$$

$$\overset{?}{\le}\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{\sqrt{2R}}{4Rrs}\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow 4R(s^2+4Rr+r^2)\overset{?}{\ge}(4R+r)(s^2+2Rr+r^2)\Leftrightarrow 8R^2-2Rr-r^2\overset{?}{\ge}s^2\to\text{true}$$

$$\because 8R^2-2Rr-r^2-s^2=2(R-2r)(2R+r)+4R^2+4Rr+3r^2-s^2\overset{\text{Euler and Gerretsen}}{\ge}0$$

$$\therefore\frac{1}{\sqrt{w_a}}+\frac{1}{\sqrt{w_b}}+\frac{1}{\sqrt{w_c}}\le\sqrt{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ \forall\,\Delta\,ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$