ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}\le m_a m_b m_c\le\frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\to(2)\text{ and,}$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\to(3)$$

$$\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{1}{64}\left(\begin{matrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}}{64}$$

$$\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3}{16}\to(m)$$

$$\therefore (m) \Rightarrow m_a m_b m_c \geq \frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}$$

$$\Leftrightarrow s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3$$
$$\geq s^4(s^2-16Rr+21r^2)$$

$$\Leftrightarrow (4R+12r)s^4 - r(60R^2+120Rr+33r^2)s^2 - r^2(4R+r)^3 \overset{(*)}{\geq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (4R+12r)(16Rr-5r^2)s^2 - r(60R^2+120Rr+33r^2)s^2$$
$$-r^2(4R+r)^3 \overset{?}{\geq} 0 \Leftrightarrow (4R^2+52Rr-93r^2)s^2 - r(4R+r)^3 \underset{(**)}{\overset{?}{\geq}} 0$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (4R^2+52Rr-93r^2)(16Rr-5r^2) - r(4R+r)^3 \overset{?}{\geq} 0$$
$$\Leftrightarrow 4r(191R^2-440Rr+116r^2) \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(191R-58r) \overset{?}{\geq} 0 \rightarrow \text{true}$$
$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore m_a m_b m_c \geq \frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2}$$

$$\textit{and also}, \text{via } (m), m_a m_b m_c \leq \frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2}$$
$$\Leftrightarrow -(11Rr-11r^2)s^4 \geq$$
$$-s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3$$

$$\Leftrightarrow (R-22r)s^4 + r(60R^2+120Rr+33r^2)s^2 + r^2(4R+r)^3 \overset{(\bullet)}{\geq} 0$$

and (•) is trivially true if $R - 22r \geq 0$ and so, we now focus on the case when :

$$R-22r<0 \text{ and then : LHS of } (\bullet) \overset{\text{Gerretsen}}{\geq} (R-22r)(4R^2+4Rr+3r^2)s^2$$
$$+r(60R^2+120Rr+33r^2)s^2 + r^2(4R+r)^3 \overset{?}{\geq} 0$$
$$\Leftrightarrow (4R^3-24R^2r+35Rr^2-33r^3)s^2 + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 0$$

and (••) is trivially true if $4R^3 - 24R^2r + 35Rr^2 - 33r^3 \geq 0$ and so,
we now focus on the case when : $4R^3 - 24R^2r + 35Rr^2 - 33r^3 < 0$ and then :

$$\text{LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq}$$
$$(4R^3-24R^2r+35Rr^2-33r^3)(4R^2+4Rr+3r^2) + r^2(4R+r)^3 \overset{?}{\geq} 0$$
$$\Leftrightarrow 16t^5-80t^4+120t^3-16t^2-15t-98 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(6t^3+2t(t^2-4)+8t^2(t-2)+16)+81\Big) \overset{?}{\geq} 0 \rightarrow \text{true}$$
$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true } \therefore m_a m_b m_c \leq \frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2} \text{ and so,}$$
$$\frac{1}{4}s^2.\sqrt{s^2-16Rr+21r^2} \leq m_a m_b m_c \leq \frac{1}{4}s^2.\sqrt{s^2-11Rr+11r^2}\ \forall\, \Delta\, ABC,$$
$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$