ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}\le\frac{13R}{8r}-\frac{1}{4}$$

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$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}.(b-c)^2\text{ and analogs}$$
$$\therefore\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}=\sum_{cyc}\left(\sqrt{\frac{as(s-a)+s(b-c)^2}{a}}.\frac{(b+c).\sqrt{(s-b)(s-c)}}{2\sqrt{bc}.\sqrt{s(s-a)(s-b)(s-c)}}\right)$$
$$=\frac{1}{2\sqrt{4Rrs}.rs}.\sum_{cyc}\left(\sqrt{(as(s-a)+s(b-c)^2)(b+c)}.\sqrt{(b+c)(s-b)(s-c)}\right)\overset{CBS}{\le}$$
$$\frac{1}{2\sqrt{4Rrs}.rs}.\sqrt{\sum_{cyc}\left((as(s-a)+s(b-c)^2)(b+c)\right)}.\sqrt{\sum_{cyc}\left((b+c)(s-b)(s-c)\right)}\to(a)$$
$$\text{Now, }\sum_{cyc}\left((as(s-a)+s(b-c)^2)(b+c)\right)$$
$$=s\sum_{cyc}\left(a(s-a)(2s-a)\right)+s\sum_{cyc}\left((2s-a)(b-c)^2\right)$$
$$=2s^2\left(s(2s)-\sum_{cyc}a^2\right)-s\left(s\sum_{cyc}a^2-\sum_{cyc}a^3\right)+4s^2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$
$$-s\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)$$
$$=2s^2\left(2s^2-2(s^2-4Rr-r^2)\right)-2s^2\left((s^2-4Rr-r^2)-(s^2-6Rr-3r^2)\right)$$
$$+4s^2(s^2-12Rr-3r^2)-2s^2(s^2-14Rr+r^2)$$

$$\Rightarrow \sum_{cyc}\Big((as(s-a)+s(b-c)^2)(b+c)\Big) = 2s^2(s^2-4Rr-7r^2) \to (m) \text{ and}$$

$$\sum_{cyc}\big((b+c)(s-b)(s-c)\big) = \sum_{cyc}\big((s-b)(s-c)(2s-a)\big)$$

$$= 2s(4Rr+r^2) - \sum_{cyc}\Big(a(-s^2+sa+bc)\Big)$$

$$= 2s(4Rr+r^2) + s^2(2s) - 2s(s^2-4Rr-r^2) - 12Rrs$$

$$\Rightarrow \sum_{cyc}\big((b+c)(s-b)(s-c)\big) = 4rs(R+r) \to (n) \therefore (a),(m),(n) \Rightarrow$$

$$\left(\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}\right)^2 \le \frac{2s^2(s^2-4Rr-7r^2).4rs(R+r)}{16Rrs.r^2s^2} \overset{?}{\le} \left(\frac{13R}{8r}-\frac{1}{4}\right)^2$$

$$= \frac{(13R-2r)^2}{64r^2} \Leftrightarrow 32(R+r)(s^2-4Rr-7r^2) \overset{?}{\underset{①}{\le}} R(13R-2r)^2$$

$$\text{Now, LHS of } ① \overset{\text{Rouche}}{\le} 32(R+r)\left(2R^2+6Rr-8r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le} R(13R-2r)^2 \Leftrightarrow 105R^3-308R^2r+68Rr^2+256r^3 \overset{?}{\ge}$$

$$64(R+r)(R-2r).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(105R^2-98Rr-128r^2) \overset{?}{\underset{②}{\ge}} 64(R+r)(R-2r).\sqrt{R^2-2Rr}$$

$$\text{Now, } 105R^2-98Rr-128r^2 = (R-2r)(105R+112r)+96r^2 \overset{\text{Euler}}{\ge} 96r^2 > 0 \text{ and}$$

$$\because R-2r \overset{\text{Euler}}{\ge} 0 \therefore \text{in order to prove } ②\text{, it suffices to prove :}$$

$$(105R^2-98Rr-128r^2)^2 > 4096(R^2-2Rr)(R+r)^2$$

$$\Leftrightarrow 6929t^4-20580t^3-4988t^2+33280t+16384 > 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \frac{1}{27}\Big((20787t^2+35266t+36437)(3t-7)^2+700880(t-2)+58715\Big) > 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \left(\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c}\right)^2 \le \left(\frac{13R}{8r}-\frac{1}{4}\right)^2 \text{ and so,}$$

$$\frac{n_a}{w_a}+\frac{n_b}{w_b}+\frac{n_c}{w_c} \le \frac{13R}{8r}-\frac{1}{4} \ \forall \Delta ABC, \text{ with equality iff } \Delta ABC \text{ is equilateral (QED)}$$