ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$r_a+\frac{R}{r}h_a\ge\sqrt{3}s$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_a+\frac{R}{r}h_a=\frac{rs(s-b)(s-c)}{(s-a)(s-b)(s-c)}+\frac{R}{r}\cdot\frac{2rs}{4Rrs}.bc=\frac{4(s-b)(s-c)}{4r}+\frac{2bc}{4r}$$

$$=\frac{a^2-(b-c)^2+2bc}{4r}\overset{?}{\ge}\sqrt{3}s\Leftrightarrow(a^2-(b-c)^2+2bc)^2\overset{?}{\ge}3.16F^2$$

$$\Leftrightarrow(a^2-(b-c)^2+2bc)^2\overset{?}{\ge}3\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$\Leftrightarrow a^4-2a^2b^2-2a^2c^2+b^4+c^4+3b^2c^2-2b^3c-2bc^3+2a^2bc\overset{?}{\ge}0$$

$$\Leftrightarrow\left((b^2+c^2)^2+a^4-2a^2(b^2+c^2)\right)+b^2c^2-2bc(b^2+c^2-a^2)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2+c^2-a^2)^2+b^2c^2-2bc(b^2+c^2-a^2)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2+c^2-a^2-bc)^2\overset{?}{\ge}0\Leftrightarrow(2bc\cos A-bc)^2\overset{?}{\ge}0\Leftrightarrow4b^2c^2\left(\cos A-\frac{1}{2}\right)^2\overset{?}{\ge}0$$

$$\rightarrow\text{true}\therefore r_a+\frac{R}{r}h_a\ge\sqrt{3}s\ \forall\,\Delta\,ABC,''=''\text{ iff }\hat{A}=60^\circ\ (QED)$$

Solution 2 by Tapas Das-India

$$\textit{We know that}\ \frac{r}{s-a}=\tan\frac{A}{2},\frac{2R}{a}=\frac{1}{\sin A}(as\ \ a=2R\sin A)=\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}}$$

$$\textit{We need to show},r_a+\frac{R}{r}h_a\ge\sqrt{3}s\ or\ \frac{r.s}{s-a}+\frac{R}{r}.\frac{2rs}{a}\ge\sqrt{3}s$$

$$or\ \frac{r}{s-a}+\frac{2R}{a}\ge\sqrt{3}\ or\ \tan\frac{A}{2}+\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}}\ge\sqrt{3}\ or$$

$$3\tan^2\frac{A}{2}+1\ge2\sqrt{3}\tan\frac{A}{2}\ or\ 3\tan^2\frac{A}{2}-2\sqrt{3}\tan\frac{A}{2}+1\ge0$$

$$or\left(\sqrt{3}\tan\frac{A}{2}-1\right)^2\ge0\ true$$

Equality holds for an equilateral triangle