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In any Δ ABC, the following relationship holds :

$$\sqrt{w_a r_a}+\sqrt{w_b r_b}+\sqrt{w_c r_c}\geq 2\sqrt[4]{8Rr^3}\left(\frac{r_a}{r+r_a}+\frac{r_b}{r+r_b}+\frac{r_c}{r+r_c}\right)$$

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$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr\cos^2\frac{A}{2}\overset{\text{via (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0\Leftrightarrow(b+c-2a)^2\overset{?}{\geq}0\rightarrow\text{true}$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\text{ and analogs}\rightarrow(m)$$

$$\text{We have : }\frac{r_a}{r+r_a}=\frac{\frac{rs}{s-a}}{\frac{rs}{s}+\frac{rs}{s-a}}=\frac{s}{s+s-a}\Rightarrow\frac{r_a}{r+r_a}=\frac{s}{b+c}\therefore\sqrt{w_a r_a}.\frac{r+r_a}{r_a}$$

$$=\frac{1}{s}.\sqrt{\frac{2bc}{b+c}.\cos\frac{A}{2}.s\tan\frac{A}{2}.(b+c)^2}=\frac{1}{s}.\sqrt{2sbc.\sin\frac{A}{2}.(b+c)}$$

$$\overset{\text{via (m)}}{\geq}\frac{1}{s}.\sqrt{2sbc.\sin\frac{A}{2}\cos\frac{A}{2}.2.\sqrt{8Rr}}=\frac{2}{s}.\sqrt{sbc.\sqrt{8Rr}.\frac{a}{4R}}=\frac{2}{s}.\sqrt{\sqrt{8Rr}.\frac{4Rrs^2}{4R}}$$

$$=\frac{2}{s}.s.\sqrt[4]{8Rr}.\sqrt[4]{r^2}\Rightarrow\sqrt{w_a r_a}.\frac{r+r_a}{r_a}\geq 2.\sqrt[4]{8Rr^3}\Rightarrow\sqrt{w_a r_a}\geq 2.\sqrt[4]{8Rr^3}.\frac{r_a}{r+r_a}$$

$$\text{and analogs}\therefore\sqrt{w_a r_a}+\sqrt{w_b r_b}+\sqrt{w_c r_c}\geq 2\sqrt[4]{8Rr^3}\left(\frac{r_a}{r+r_a}+\frac{r_b}{r+r_b}+\frac{r_c}{r+r_c}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)