ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$s^2 \geq \frac{27Rr}{2} \cdot \frac{(a+b)^2}{4ab}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$s = \frac{a+b+c}{2}, \frac{abc}{4s} = Rr$$

$$We\ need\ to\ show: s^2 \geq \frac{27Rr}{2} \cdot \frac{(a+b)^2}{4ab}\ or\ s^2 \geq \frac{27}{2} \cdot \frac{abc}{4s} \cdot \frac{(a+b)^2}{4ab}$$

$$or 32s^3 \geq 27\,c\,(a+b)^2\ or\ 32\left(\frac{a+b+c}{2}\right)^3 \geq 27\,c(a+b)^2$$

$$or\ 32\frac{(a+b+c)^3}{8} \geq 27\,c(a+b)^2\ \ or\ 4(a+b+c)^3 \geq 27\,c(a+b)^2$$

$$true\ since: 4(a+b+c)^3 = 4\left(c + \frac{a+b}{2} + \frac{a+b}{2}\right)^3 \overset{AM-GM}{\geq} 4\left(3\sqrt[3]{\frac{c(a+b)^2}{4}}\right)^3 = 27c(a+b)^2$$

Equality holds for an equilateral triangle.