ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{w_a+h_a+r_a}{R+r+r_a}\le\frac{h_a}{2r}$$

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Solution by Tapas Das-India

$$\frac{w_a}{h_a}=\frac{2\sqrt{bcs(s-a)}}{b+c}\frac{2R}{bc}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2s-a)}=\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}(2(s-a)+a)}\overset{AM-GM}{\le}$$

$$\le\frac{4R\sqrt{s(s-a)}}{\sqrt{bc}2\sqrt{2(s-a).a}}=\frac{2R\sqrt{s}}{\sqrt{2abc}}=\frac{2R\sqrt{s}}{\sqrt{8Rrs}}=\sqrt{\frac{R}{2r}}\ (1)$$

$$We\ need\ to\ show:$$

$$\frac{w_a+h_a+r_a}{R+r+r_a}\le\frac{h_a}{2r}\ \ or\ \frac{w_a+h_a+r_a}{h_a}\le\frac{R+r+r_a}{2r}\ \ or\ 1+\frac{w_a+r_a}{h_a}\le\frac{R}{2r}+\frac{1}{2}+\frac{r_a}{2r}$$

$$or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{r_a}{2r}-\frac{1}{2}\ \ or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{rs}{2r(s-a)}-\frac{1}{2}$$

$$or\ \frac{w_a}{h_a}+\frac{r_a}{h_a}\le\frac{R}{2r}+\frac{a}{2(s-a)}$$

$$or\ \sqrt{\frac{R}{2r}}+\frac{F}{s-a}\frac{a}{2F}\le\frac{R}{2r}+\frac{a}{2(s-a)}\ \ or\ \sqrt{\frac{R}{2r}}+\frac{a}{2(s-a)}\le\frac{R}{2r}+\frac{a}{2(s-a)}\ \ or$$

$$\sqrt{\frac{R}{2r}}\le\frac{R}{2r}\ \ or\ \frac{R}{2r}\le\frac{R^2}{4r^2}\ \ or\ R\ge 2r\ Euler$$

Equality holds for an equilateral triangle.