ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{1}{m_a w_b w_c}+\frac{1}{m_b w_c w_a}+\frac{1}{m_c w_a w_b}\le\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{h_a+h_b+h_c}{4r_a r_b r_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } m_a\ge w_a \text{ and analogs } \therefore \frac{1}{m_a w_b w_c}+\frac{1}{m_b w_c w_a}+\frac{1}{m_c w_a w_b}\le\frac{3}{w_a w_b w_c}$$

$$=\frac{3(s^2+2Rr+r^2)}{16Rr^2s^2}\overset{?}{\le}\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{h_a+h_b+h_c}{4r_a r_b r_c}\right)=\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2}+\frac{s^2+4Rr+r^2}{8Rrs^2}\right)$$

$$=\frac{\sqrt{3}}{s}\left(\frac{2Rs^2+r(s^2+4Rr+r^2)}{8Rr^2s^2}\right)$$

$$\Leftrightarrow\frac{9(s^2+2Rr+r^2)^2}{4}\overset{?}{\le}\frac{3\left(2Rs^2+r(s^2+4Rr+r^2)\right)^2}{s^2}\Leftrightarrow$$

$$-3s^6+(16R^2+4Rr-2r^2)s^4+r^2(52R^2+36Rr+5r^2)s^2+4r^4(4R+r)^2\ \boxed{\overset{?}{\ge}\ (1)}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(a)}{\le}0$$

$$\Rightarrow -3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\ge 0 \text{ and so,}$$

in order to prove (1), it suffices to prove : LHS of (1) $\ge -3s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$

$$\Leftrightarrow (R^2-14Rr+r^2)s^4+r(48R^3+49R^2r+18Rr^2+2r^3)s^2+r^4(4R+r)^2\ \boxed{\overset{?}{\ge}\ (2)}\ 0$$

We note that (2) is trivially true if : $R^2-14Rr+r^2\ge 0$ and so we now focus on the case when : $R^2-14Rr+r^2<0$ and then :

$$(R^2-14Rr+r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{\text{via }(a)}{\ge}0 \text{ and so,}$$

in order to prove (2), it suffices to prove :

$$\text{LHS of (2)}\ge(R^2-14Rr+r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow(4R^3+12R^2r-229Rr^2+66r^3)s^2$$

$$-r(64R^4-848R^3r-596R^2r^2-135Rr^3-10r^4)\ \boxed{\overset{(3)}{\ge}}\ 0$$

Case 1 $4R^3+12R^2r-229Rr^2+66r^3\ge 0$ and then : LHS of (3) $\overset{\text{Gerretsen}}{\ge}(4R^3+12R^2r-229Rr^2+66r^3)(16Rr-5r^2)$

$$-r(64R^4 - 848R^3r - 596R^2r^2 - 135Rr^3 - 10r^4) \overset{?}{\geq} 0$$

$$\Leftrightarrow 255t^3 - 782t^2 + 584t - 80 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(255t^2 - 272t + 40) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \text{ is true}$$

Case 2 $4R^3 + 12R^2r - 229Rr^2 + 66r^3 < 0$ *and* then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$ $(4R^3 + 12R^2r - 229Rr^2 + 66r^3)(4R^2 + 4Rr + 3r^2)$

$$-r(64R^4 - 848R^3r - 596R^2r^2 - 135Rr^3 - 10r^4) \overset{?}{\geq} 0$$

$$\Leftrightarrow 4t^5 - 2t^3 - 5t^2 - 72t + 52 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(4t^4 + 8t^3 + 14t^2 + 23t - 26) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ③ is true $\therefore$ combining both cases, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\forall \Delta ABC \therefore \frac{1}{m_a w_b w_c} + \frac{1}{m_b w_c w_a} + \frac{1}{m_c w_a w_b} \leq$$

$$\frac{\sqrt{3}}{s}\left(\frac{1}{4r^2} + \frac{h_a + h_b + h_c}{4r_a r_b r_c}\right) \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$