ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{m_a}{a}\le\sum_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{s}{2r}\overset{?}{\ge}\sum_{cyc}\frac{m_a}{a}\Leftrightarrow\frac{s}{2r}\overset{?}{\ge}\frac{1}{4Rrs}\sum_{cyc}bcm_a\Leftrightarrow(2Rs^2)^2\overset{?}{\ge}\left(\sum_{cyc}bcm_a\right)^2$$

$$=\sum_{cyc}\left(b^2c^2\left(\frac{2b^2+2c^2-a^2}{4}\right)\right)+2\sum_{cyc}(bc.ca.m_am_b)$$

$$\Leftrightarrow16R^2s^4\overset{?}{\ge}\sum_{cyc}\left(b^2c^2\left(2\sum_{cyc}a^2-3a^2\right)\right)+32Rrs\sum_{cyc}cm_am_b$$

$$\Leftrightarrow16R^2s^4\overset{?}{\underset{(1)}{\ge}}4(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$-144R^2r^2s^2+32Rrs\sum_{cyc}cm_am_b$$

$$\text{Now, }m_am_b\overset{?}{\le}\frac{2c^2+ab}{4}\Leftrightarrow\left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right)\overset{?}{\le}\frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2\overset{?}{\ge}0$$

$$\Leftrightarrow(a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\ge}0$$

$$\Leftrightarrow(a-b)^2(a+b+c)(a+b-c)\overset{?}{\ge}0\to\text{true}\Rightarrow m_am_b\le\frac{2c^2+ab}{4}\text{ and analogs}$$

$$\therefore\text{RHS of }(1)\le4(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-144R^2r^2s^2$$

$$+32Rrs\sum_{cyc}\left(c.\frac{2c^2+ab}{4}\right)\overset{?}{\le}16R^2s^4$$

$$\Leftrightarrow(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-36R^2r^2s^2$$

$$+8Rrs(4(s^2-6Rr-3r^2)+12Rrs)\overset{?}{\le}4R^2s^4$$

$$\Leftrightarrow s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3\overset{?}{\underset{(2)}{\le}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge0$ and $s^2-(m+n)\le0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Rightarrow s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2\le 0$$

$\therefore$ in order to prove ②, it suffices to prove :

$$s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3\overset{?}{\le}$$
$$s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2$$

$$\Leftrightarrow (16R-5r)s^4-(64R^3+60R^2r+28Rr^2+2r^3)s^2-r^2(4R+r)^3\underset{③}{\overset{?}{\le}}0$$

$$\text{Again, LHS of ③}\overset{\text{Gerretsen}}{\le}$$
$$\left((16R-5r)(4R^2+4Rr+3r^2)-(64R^3+60R^2r+28Rr^2+2r^3)\right)s^2$$
$$-r^2(4R+r)^3\overset{?}{\le}0\Leftrightarrow(16R-5r)s^2\underset{④}{\overset{?}{\le}}(4R+r)^3$$

$$\text{Finally, LHS of ④}\overset{\text{Gerretsen}}{\le}(16R-5r)(4R^2+4Rr+3r^2)\overset{?}{\le}(4R+r)^3$$

$$\Leftrightarrow 4r(R-2r)^2\overset{?}{\ge}0\to\text{true}\Rightarrow ④\Rightarrow ③\Rightarrow ②\Rightarrow ①\text{ is true}\therefore 2\sum_{cyc}\frac{m_a}{a}\le\frac{s}{r}\to(\mathrm{m})$$

$$\text{We have}:\sum_{cyc}\frac{r_b+r_c}{a}=\sum_{cyc}\frac{4R\cos^2\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}=s.\sum_{cyc}\frac{1}{r_a}\Rightarrow\sum_{cyc}\frac{r_b+r_c}{a}=\frac{s}{r}\to(*)$$

$$\text{and also},\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{4R\cos^2\frac{A}{2}}{a}=\frac{64R^3.\frac{s^2}{16R^2}}{4Rrs}\Rightarrow\prod_{cyc}\frac{r_b+r_c}{a}=\frac{s}{r}\to(**)$$

$$\text{Moreover},\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\prod_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.(s-a)}{ra}}=\sqrt{\frac{64R^3.\frac{s^2}{16R^2}.r^2s}{r^3.4Rrs}}$$

$$\Rightarrow\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\frac{s}{r}\to(***)\text{ and finally},\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.(s-a)}{ra}}$$

$$=\sum_{cyc}\sqrt{\frac{4R\cos^2\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}}=s\sum_{cyc}\sqrt{\frac{1}{r_a^2}}\Rightarrow\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}\frac{s}{r}\to(****)$$

$$\therefore(\mathrm{m}),(*),(**),(***)\text{ and }(****)\Rightarrow$$

$$2\sum_{cyc}\frac{m_a}{a}\le\sum_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\frac{r_b+r_c}{a}=\prod_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}=\sum_{cyc}\sqrt{\frac{r_b+r_c}{r_a-r}}$$

$\forall\,\Delta\,ABC$," =" iff $\Delta\,ABC$ is equilateral (QED)