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In any Δ ABC, the following relationship holds :

$$p_a p_b p_c \geq s^2 r \sqrt{\frac{7R-4r}{5R}}$$

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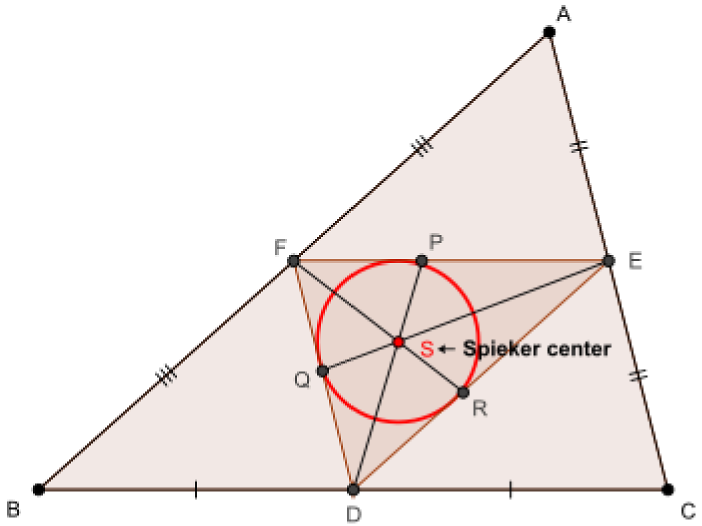

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \boxed{\overset{(i)}{=}} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\ \frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\ \boxed{\overset{(**)}{=}}\ \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\ \boxed{\overset{(ii)}{=}}\ \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\text{We have}: \prod_{cyc}(2s+a) = 8s^3+4s^2\sum_{cyc}a+2s\sum_{cyc}ab+4Rrs =$$

$$8s^3+4s^2.2s+2s(s^2+4Rr+r^2)+4Rrs \Rightarrow \prod_{cyc}(2s+a) \overset{(\bullet\bullet)}{=} 2s(9s^2+6Rr+r^2)$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{cyc}a^3-abc+2a.2bc\cos A = 2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) = 2s(Q+8Rr\cos A) \text{ and analogs}$$

$$(Q = s^2-8Rr-3r^2) \Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$= 8s^3\left(Q^3+Q^2.8Rr\sum_{cyc}\cos A+Q.64R^2r^2.\sum_{cyc}\cos B\cos C+512R^3r^3\prod_{cyc}\cos A\right)$$

$$= 8s^3\left(\begin{gathered}Q^3+Q^2.8Rr.\frac{R+r}{R}+Q.32R^2r^2.\left(\left(\frac{R+r}{R}\right)^2-\left(3-\frac{2(s^2-4Rr-r^2)}{4R^2}\right)\right)\\ +512R^3r^3.\frac{s^2-(2R+r)^2}{4R^2}\end{gathered}\right)$$

$$\Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right) \overset{(\bullet\bullet\bullet)}{=}$$

$$8s^3\left(\begin{gathered}(s^2-8Rr-3r^2)^3+(s^2-8Rr-3r^2)^2.8r(R+r)+\\ (s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2)+128Rr^3(s^2-(2R+r)^2)\end{gathered}\right)$$

$$\therefore (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \Rightarrow \prod_{cyc}p_a^2 =$$

$$\frac{8s^3.8s^3\left(\begin{gathered}(s^2-8Rr-3r^2)^3+(s^2-8Rr-3r^2)^2.8r(R+r)+\\ (s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2)+128Rr^3(s^2-(2R+r)^2)\end{gathered}\right)}{4s^2(9s^2+6Rr+r^2)^2} \overset{?}{\geq} s^4r^2.\frac{7R-4r}{5R}$$

$$\Leftrightarrow 80Rs^6-r(1280R^2-633Rr-324r^2)s^4-r^3(3316R^2+3934Rr-72r^2)s^2$$

$$-r^4(252R^3+1220R^2r+199Rr^2-4r^3) \overset{?}{\underset{①}{\geq}} 0 \text{ and } \because$$

$$P = 80R(s^2-16Rr+5r^2)^3+r(2560R^2-567Rr+324r^2)(s^2-16Rr+5r^2)^2+$$

$$4r^2(5120R^3-2165R^2r+1526Rr^2-792r^3)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } ①, \text{it suffices to prove : LHS of } ① \overset{?}{\geq} P$$

$$\Leftrightarrow 1585t^3-480t^2-6348t+1936 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1585t^2+2690t-968) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow ① \text{ is true} \therefore \prod_{cyc}p_a^2 \geq s^4r^2.\frac{7R-4r}{5R} \Rightarrow p_ap_bp_c \geq s^2r.\sqrt{\frac{7R-4r}{5R}}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$