ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$n_a \le h_a + 4R\left(\frac{b-c}{a}\right)^2 \le h_a + \frac{16}{3}(R-2r)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2$$
$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (i)$$

$$\text{Now, } n_a \overset{?}{\ge} \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\ge} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\ge} \left(\frac{b^2-bc+c^2}{bc}\right)^2 - 1$$
$$= \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \overset{?}{\ge} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\overset{\text{via (1)}}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}$$
$$\overset{?}{\ge} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}.\frac{b^2c^2}{4R^2} \Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\ge} \frac{(b-c)^2(b^2+c^2)}{4R^2}$$
$$\Leftrightarrow \frac{s^2}{a^2} \overset{?}{\ge} \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \ge 0) \Leftrightarrow 4R^2s^2 \overset{?}{\ge} a^2b^2 + c^2a^2 \to \text{true}$$
$$(\text{strict inequality}) \because 4R^2s^2 \overset{\text{Goldstone}}{\ge} \sum_{\text{cyc}} a^2b^2 > a^2b^2 + c^2a^2 \therefore n_a \ge \frac{b^2-bc+c^2}{2R}$$

$$\Rightarrow n_a - h_a = \frac{n_a^2 - h_a^2}{n_a + h_a} \le \frac{s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{\frac{b^2+c^2}{2R}}$$
$$\le \frac{\frac{s^2}{a^2}(b-c)^2}{\frac{(b+c)^2}{4R}} \le \frac{\frac{s^2}{a^2}(b-c)^2}{\frac{s^2}{4R}}\ (\because b+c = s+s-a > s) \Rightarrow n_a - h_a \le 4R\left(\frac{b-c}{a}\right)^2$$
$$\Rightarrow \boxed{n_a \le h_a + 4R\left(\frac{b-c}{a}\right)^2} \overset{?}{\le} h_a + \frac{16}{3}(R-2r) \Leftrightarrow \frac{4}{3}(R-2r).a^2 \overset{?}{\ge} R(b-c)^2$$
$$\Leftrightarrow \frac{4}{3}.R\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right).16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}$$
$$\overset{?}{\ge} R.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$
$$\Leftrightarrow \frac{4}{3}\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right)\left(1 - \sin^2\frac{A}{2}\right) \overset{?}{\ge} 1 - \cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 3\cos^2\frac{B-C}{2}-16\left(x-x^3\right)\cos\frac{B-C}{2}+1+12x^2-16x^4 \overset{?}{\underset{①}{\geq}} 0\ \left(x=\sin\frac{A}{2}\right)$$

Now, LHS of ① is a quadratic polynomial in $\cos\frac{B-C}{2}$ with discriminant $=$

$$256\left(x-x^3\right)^2-12\left(1+12x^2-16x^4\right)=256x^6-320x^4+112x^2-12$$

$$=4\left(4x^2-1\right)^2\left(4x^2-3\right)\leq 0 \text{ iff } x\leq\frac{\sqrt{3}}{2}$$ and so, when $x\leq\frac{\sqrt{3}}{2}$, discriminant ≤ 0

$\Rightarrow$ LHS of ① $\geq 0 \Rightarrow$ ① is true and we now focus on the scenario :

when $x>\frac{\sqrt{3}}{2}$ and then, in order to prove ①, it suffices to prove : $\cos\frac{B-C}{2}>$

$$\frac{8\left(x-x^3\right)+\left(4x^2-1\right).\sqrt{4x^2-3}}{3}\ \left(\because x>\frac{\sqrt{3}}{2}>\frac{1}{2}\Rightarrow 4x^2-1>0\right)$$

and $\because \cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}\overset{\frac{b+c}{a}>1}{>} x$ $\therefore$ it suffices to prove : $x>$

$$\Leftrightarrow\left(8x^3-5x\right)^2>\left(4x^2-3\right)\left(4x^2-1\right)^2\ \left(\because x>\frac{\sqrt{3}}{2}\Rightarrow 8x^2>6>5\Rightarrow 8x^3>5x\right)$$

$\Leftrightarrow 3-3x^2>0\Rightarrow 1>x^2\rightarrow$ true $\Rightarrow$ ① is true and combining both cases,

① is true $\forall\,\Delta\,ABC \Rightarrow$ $\boxed{h_a+4R\left(\frac{b-c}{a}\right)^2\leq h_a+\frac{16}{3}(R-2r)}$ and so,

$$n_a\leq h_a+4R\left(\frac{b-c}{a}\right)^2\leq h_a+\frac{16}{3}(R-2r)\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)