ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\ge\frac{3}{\sqrt[3]{s^2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\overset{A-G}{\ge}3.\sqrt[3]{\frac{1}{w_ar_ah_a}}=3.\sqrt[3]{\frac{b+c}{2bc\cos\frac{A}{2}.\frac{s\tan\frac{A}{2}.2rs}{4R\cos^2\frac{A}{2}.\tan\frac{A}{2}}}}\overset{?}{\ge}\frac{3}{\sqrt[3]{s^2r}}$$

$$\Leftrightarrow\frac{Ra(b+c)}{4Rrs\cos\frac{A}{2}.\sec^2\frac{A}{2}}\overset{?}{\ge}1$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\sin\frac{A}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\ge}4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\sec\frac{A}{2}$$

$$\Leftrightarrow\cos^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\ge}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$\Leftrightarrow\cos\frac{B-C}{2}-\sin^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\ge}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow\cos\frac{B-C}{2}-\cos^2\frac{B-C}{2}+\sin^2\frac{A}{2}-\sin^2\frac{A}{2}\cos\frac{B-C}{2}\overset{?}{\ge}0$$

$$\Leftrightarrow\left(\cos\frac{B-C}{2}+\sin^2\frac{A}{2}\right)\left(1-\cos\frac{B-C}{2}\right)\overset{?}{\ge}0\to\text{true}\because 0<\cos\frac{B-C}{2}\le 1$$

$$\therefore\frac{1}{w_a}+\frac{1}{r_a}+\frac{1}{h_a}\ge\frac{3}{\sqrt[3]{s^2r}}\ \forall\,\Delta\,ABC\ (QED)$$