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In any Δ ABC the following relationship holds :

$$\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}}\geq\frac{\sqrt{3}}{2}\cdot\sum_{sym}\sqrt{\frac{a}{b}}$$

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$$\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\prod_{cyc}\sqrt{\frac{2rs(b+c).as}{bc.rs(2s-a)}}=\sqrt{\frac{8.4Rrs.s^3}{16R^2r^2s^2}}$$

$$\Rightarrow\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\sqrt{\frac{2s^2}{Rr}}\rightarrow(i)$$

$$\sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\sum_{cyc}\sqrt{\frac{2rs(b+c).as}{bc.rs(2s-a)}}=\sqrt{\frac{2s}{4Rrs}}.\sum_{cyc}a\Rightarrow\sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\sqrt{\frac{2s^2}{Rr}}\rightarrow(ii)$$

$$\prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}}=\sqrt{\frac{2r^2s^2}{R}\cdot\frac{1}{r^3}}\Rightarrow\prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}}=\sqrt{\frac{2s^2}{Rr}}\rightarrow(iii)$$

$$\text{Now, }\frac{\sqrt{3}}{2}.\sum_{sym}\sqrt{\frac{a}{b}}=\frac{\sqrt{3}}{2}.\sum_{cyc}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{4Rrs}}\sum_{cyc}\left(\sqrt{a(b+c)}.\sqrt{b+c}\right)$$

$$\overset{CBS}{\leq}\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{4Rrs}}.\sqrt{2\sum_{cyc}ab}.\sqrt{4s}\leq\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{Rr}}.\sqrt{\frac{2}{3}.4s^2}\therefore\frac{\sqrt{3}}{2}.\sum_{sym}\sqrt{\frac{a}{b}}\leq\sqrt{\frac{2s^2}{Rr}}\rightarrow(m)$$

$$\therefore(i),(ii),(iii),(m)\Rightarrow\prod_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\sum_{cyc}\sqrt{\frac{h_b+h_c}{h_a-r}}=\prod_{cyc}\sqrt{\frac{h_bh_c}{h_ar}}\geq\frac{\sqrt{3}}{2}.\sum_{sym}\sqrt{\frac{a}{b}}$$

$\forall\,\Delta$ ABC," = " iff Δ ABC is equilateral (QED)