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In any Δ ABC the following relationship holds :

$$\left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a \geq \sum_{cyc} m_a+4050\sum_{cyc} w_a$$

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$$\sum_{cyc} w_a=\sum_{cyc}\left(2\sqrt{bc}.\frac{\sqrt{s(s-a)}}{b+c}\right) \overset{CBS}{\leq} 2\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}$$

$$\overset{A-G}{\leq} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{s(s-a)}{bc}}=\sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}=\sqrt{\sum_{cyc} bc}.\sqrt{\frac{4R+r}{2R}}$$

$$=\sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \Rightarrow \sum_{cyc} w_a \leq \sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \rightarrow (a)$$

$$\text{Now, }\left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a$$

$$=\left(\frac{R}{r}-1\right)\left(\sum_{cyc} h_a\right)+2025\left(\sum_{cyc} h_a+\sum_{cyc} r_a\right) \overset{A-G}{\geq}$$

$$\left(\frac{R}{2r}-1+\frac{R}{2r}\right)\left(\sum_{cyc} h_a\right)+4050.\sqrt{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a\right)} \overset{\substack{\text{Euler + Panaitopol}\\ \text{and}\\ \text{via }(a)}}{\geq}$$

$$\sum_{cyc} m_a+4050\sum_{cyc} w_a \therefore \left(\frac{R}{r}+2024\right)\left(\sum_{cyc} h_a\right)+2025\sum_{cyc} r_a \geq$$

$$\sum_{cyc} m_a+4050\sum_{cyc} w_a \;\forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$