ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{g_a}{h_a}+\frac{g_b}{h_b}+\frac{g_c}{h_c}\le\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\text{and } b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c) \text{ and via summation, we get :}$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)=$$
$$2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+$$
$$a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2=$$
$$2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2\overset{(**)}{=}s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\text{Via } (*) \text{ and } (**),\ g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)$$
$$=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}=(s-a)^2+\frac{4r^2s}{a}=(s-a)^2+2rh_a$$
$$\Rightarrow\frac{g_a^2}{h_a}=\frac{a(s-a)^2}{2rs}+2r \text{ and analogs} \Rightarrow\sum_{cyc}\frac{g_a^2}{h_a}=\frac{1}{2rs}.\sum_{cyc}a(s^2-2sa+a^2)+6r$$
$$=\frac{1}{2rs}.\left(s^2(2s)-4s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)\right)+6r$$

$$=\frac{4Rrs-2r^2s}{2rs}+6r\therefore\sum_{cyc}\frac{g_a^2}{h_a}=2R+5r\therefore\frac{g_a}{h_a}+\frac{g_b}{h_b}+\frac{g_c}{h_c}\overset{CBS}{\leq}\sqrt{\sum_{cyc}\frac{g_a^2}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}}$$

$$=\sqrt{\frac{2R+5r}{r}}\overset{?}{\leq}\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}=\sum_{cyc}\sqrt{\frac{s\tan\frac{A}{2}.4R\cos^2\frac{A}{2}\tan\frac{A}{2}}{2rs}}=\sqrt{\frac{2R}{r}}.\sum_{cyc}\sin\frac{A}{2}$$

$$\Leftrightarrow\left(\frac{2R}{r}\right)\left(\sum_{cyc}\sin^2\frac{A}{2}+2\left(\prod_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\mathrm{cosec}\frac{A}{2}\right)\right)\underset{(1)}{\overset{?}{\geq}}\frac{2R+5r}{r}$$

$$\because\sum_{cyc}\mathrm{cosec}\frac{A}{2}\overset{Jensen}{\geq}6\therefore \text{LHS of (1)}\geq\left(\frac{2R}{r}\right)\left(\frac{2R-r}{2R}+\frac{r}{2R}.6\right)=\frac{2R+5r}{r}$$

$$\Rightarrow \text{(1) is true}\Rightarrow\frac{g_a}{h_a}+\frac{g_b}{h_b}+\frac{g_c}{h_c}\leq\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}}\ \forall\,\Delta\,ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)