ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{b^2+c^2}{b+c}\cos\frac{A}{2}\le n_a$$

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$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a},\qquad g_a^2=s(s-a)-\frac{(s-a)(b-c)^2}{a}$$

$$m_a^2=s(s-a)+\frac{(b-c)^2}{4}\ \text{or}\ 4m_a^2=4s(s-a)+(b-c)^2$$

Using above result we get:

$$n_a^2+g_a^2=(b-c)^2+2s(s-a)$$

$$n_aw_a\overset{n_a\ge m_a}{\ge}m_aw_a\overset{\text{Known inequality } m_a\ge\frac{b+c}{2}\cos\frac{A}{2}}{\ge}\frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cos\frac{A}{2}=$$

$$=bc\cos^2\frac{A}{2}=s(s-a)$$

$$(n_a+w_a)^2=n_a^2+g_a^2+2\,n_aw_a\ge(b-c)^2+2s(s-a)+2\,m_aw_a\ge$$

$$\ge(b-c)^2+2s(s-a)+2s(s-a)=$$

$$=4s(s-a)+(b-c)^2=\ 4m_a^2\ \text{or},n_a+w_a\ge 2m_a\ (1)$$

$$\frac{b^2+c^2}{b+c}\cos\frac{A}{2}=\left((b+c)-\frac{2bc}{b+c}\right)\cos\frac{A}{2}=(b+c)\cos\frac{A}{2}-\frac{2bc}{b+c}\cos\frac{A}{2}\le$$

$$\overset{m_a\ge\frac{b+c}{2}\cos\frac{A}{2}}{\le}2\,m_a-w_a\overset{(1)}{\le}n_a+w_a-w_a=n_a$$