ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$m_a h_a \le w_a\left(\frac{n_a+h_a}{2}\right)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}(b+c)} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a \Leftrightarrow$$

$$4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\le} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \le 1 \to \text{true} \therefore g_a \le w_a \to (1)$$

$$\text{Now, } an_a^2.ag_a^2 \overset{?}{\ge} a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\left(\begin{matrix}b^2(s-b)+c^2(s-c)\\-a(s-b)(s-c)\end{matrix}\right) \underset{(*)}{\overset{?}{\ge}} a^2s^2(s-a)^2$$

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z \Rightarrow a=y+z,$$
$$b=z+x \text{ and } c=x+y \text{ and via such substitutions, } (*) \Leftrightarrow$$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$
$$\ge x^2(y+z)^2(x+y+z)^2 \Leftrightarrow xy^2+xz^2+y^3+z^3 \ge 2xyz+yz(y+z)$$
$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \ge 0 \to \text{true} \Rightarrow (*) \text{ is true} \Rightarrow n_a g_a \ge s(s-a) \to (2)$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2+a(s-b)(s-c)$$
$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2+a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$
$$(b^2+c^2)(2s-b-c) = an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2) = 2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$
$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2)+a^2-(b-c)^2$$
$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2 = 2(n_a^2+g_a^2)$$
$$\Rightarrow 2(b-c)^2+4s(s-a) = 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 = (b-c)^2+2s(s-a)$$

$$\Rightarrow n_a^2+g_a^2+2n_a g_a \overset{\text{via } (2)}{\ge} (b-c)^2+4s(s-a) \Rightarrow (n_a+g_a)^2 \ge 4m_a^2$$

$$\Rightarrow m_a \le \frac{n_a+g_a}{2} \Rightarrow m_a h_a \le h_a.\frac{n_a+g_a}{2} \overset{?}{\le} w_a\left(\frac{n_a+h_a}{2}\right)$$

$$\Leftrightarrow n_a(w_a-h_a)+h_a(w_a-g_a) \overset{?}{\ge} 0 \to \text{true via } (1)$$

$$\therefore m_a h_a \le w_a\left(\frac{n_a+h_a}{2}\right) \ \forall \Delta ABC, '' ='' \text{ iff } b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $n_a \geq m_a$ and $m_a w_a \geq s(s-a)$, then we have

$$(n_a + w_a)^2 \geq n_a{}^2 + 2m_a w_a + w_a{}^2 \geq s\left(s - a + \frac{(b-c)^2}{a}\right) + 2s(s-a) + \frac{4bcs(s-a)}{(b+c)^2}$$

$$= 4s(s-a) + \frac{s(b-c)^2}{a} - \frac{s(s-a)(b-c)^2}{(b+c)^2}$$

$$= 4s(s-a) + (b-c)^2 + \frac{(s-a)^2(4s-a)(b-c)^2}{(b+c)^2}$$

$$\geq 4s(s-a) + (b-c)^2 = 2(b^2+c^2) - a^2 = 4m_a{}^2 \Rightarrow n_a + w_a \geq 2m_a.$$

Therefore

$$m_a h_a \leq \frac{(n_a + w_a)h_a}{2} = \frac{h_a n_a + w_a h_a}{2} \leq \frac{w_a n_a + w_a h_a}{2} = w_a\left(\frac{n_a + h_a}{2}\right).$$

Equality holds iff $b = c$.