ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\prod_{cyc}\sqrt{w_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\sqrt{g_a^2-h_a^2}\cdot\prod_{cyc}\sqrt{n_a^2-h_a^2}\le 4Rs^2(R-2r)^3$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)$$
$$=an_a^2+a(s-b)(s-c)\text{ and }b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)\text{ and}$$
$$\text{via summation, we get : }(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)}{bc}$$
$$=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2\overset{(**)}{=}s(s-a)+\frac{s}{a}(b-c)^2$$
$$\text{Via }(*)\text{ and }(**),\ g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)^2+\frac{(s-a)(a^2-(b-c)^2)}{a}=s(s-a)-\frac{(s-a)(b-c)^2}{a}$$
$$\Rightarrow g_a^2-h_a^2=s(s-a)-\frac{(s-a)(b-c)^2}{a}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$
$$=\frac{(s-a)^2}{a^2}.(b-c)^2\Rightarrow\sqrt{g_a^2-h_a^2}=\frac{s-a}{a}.|b-c|\text{ and analogs}\rightarrow (1)$$
$$\text{Via }(**),\ n_a^2-h_a^2=s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$
$$=\frac{s^2}{a^2}.(b-c)^2\Rightarrow\sqrt{n_a^2-h_a^2}=\frac{s}{a}.|b-c|\text{ and analogs}\rightarrow (2)$$
$$\text{Also, }m_a^2-h_a^2=s(s-a)+\frac{(b-c)^2}{4}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{4s^2-4sa+a^2}{4a^2}.(b-c)^2=\frac{(b+c)^2}{4a^2}.(b-c)^2$$

$$\Rightarrow\sqrt{m_a^2-h_a^2}=\frac{b+c}{2a}.|b-c|\ \text{and analogs}\rightarrow ③$$

$$\text{Finally}, w_a^2-h_a^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{4s^2(s-a)^2}{a^2(b+c)^2}.(b-c)^2\Rightarrow\sqrt{w_a^2-h_a^2}=\frac{2s(s-a)}{a(b+c)}.|b-c|\ \text{and analogs}\rightarrow ④$$

$$\text{We have}: R-2r\overset{?}{\geq}\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\overset{?}{\geq}\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\overset{?}{\geq}\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\overset{?}{\geq}\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\overset{?}{\geq}4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\overset{?}{\geq}4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\overset{?}{\geq}0\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\overset{?}{\geq}0$$

$$\rightarrow \text{true}\Rightarrow 4R(R-2r)\geq(b-c)^2\ \text{and analogs}\rightarrow (m)\ \text{and via}\ ③\ \text{and}\ ④,$$

$$\prod_{cyc}\sqrt{w_a^2-h_a^2}.\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\left(\frac{b+c}{2a}.|b-c|\right).\prod_{cyc}\left(\frac{2s(s-a)}{a(b+c)}.|b-c|\right)$$

$$=\frac{s^3.r^2s}{16R^2r^2s^2}.\prod_{cyc}(b-c)^2\Rightarrow\prod_{cyc}\sqrt{w_a^2-h_a^2}.\prod_{cyc}\sqrt{m_a^2-h_a^2}=\frac{s^2}{16R^2}.\prod_{cyc}(b-c)^2$$

$$\rightarrow (i)\ \text{and via}\ ①\ \text{and}\ ②,\prod_{cyc}\sqrt{g_a^2-h_a^2}.\prod_{cyc}\sqrt{n_a^2-h_a^2}=$$

$$\prod_{cyc}\left(\frac{s-a}{a}.|b-c|\right).\prod_{cyc}\left(\frac{s}{a}.|b-c|\right)=\frac{s^2.r^2s^2}{16R^2r^2s^2}.\prod_{cyc}(b-c)^2$$

$$\Rightarrow\prod_{cyc}\sqrt{g_a^2-h_a^2}.\prod_{cyc}\sqrt{n_a^2-h_a^2}=\frac{s^2}{16R^2}.\prod_{cyc}(b-c)^2\rightarrow (ii)$$

$$\therefore (ii)\ \text{and}\ (ii)\Rightarrow\prod_{cyc}\sqrt{w_a^2-h_a^2}.\prod_{cyc}\sqrt{m_a^2-h_a^2}=\prod_{cyc}\sqrt{g_a^2-h_a^2}.\prod_{cyc}\sqrt{n_a^2-h_a^2}$$

$$=\frac{s^2}{16R^2}.\prod_{cyc}(b-c)^2\overset{\text{via }(m)}{\leq}\frac{s^2}{16R^2}.64R^3(R-2r)^3=4Rs^2(R-2r)^3\ \forall\ \Delta ABC,$$

$$''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$