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In any ΔABC the following relationship holds :

$$\sum_{cyc}\left(\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}\right)\le 5+\frac{2R}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a \le g_a \le w_a \le m_a \le p_a \le n_a$ (see [1, pp. 2]), then we have

$$\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}=\frac{n_a}{h_a}+\frac{g_a}{h_a}+\frac{m_a}{w_a}-n_a\left(\frac{1}{h_a}-\frac{1}{p_a}\right)\le\frac{n_a}{h_a}+\frac{g_a}{h_a}+\frac{m_a}{w_a}-p_a\left(\frac{1}{h_a}-\frac{1}{p_a}\right)$$

$$=\frac{n_a}{h_a}+1+\frac{g_a}{h_a}+\frac{m_a}{w_a}-\frac{p_a}{h_a}\le\frac{n_a}{h_a}+1+\frac{g_a}{h_a}+\frac{m_a}{g_a}-\frac{m_a}{h_a}=\frac{n_a}{h_a}+2-\left(\frac{m_a}{g_a}-1\right)\left(\frac{g_a}{h_a}-1\right)\le\frac{n_a}{h_a}+2,$$

then

$$\sum_{cyc}\left(\frac{g_a}{h_a}+\frac{m_a}{w_a}+\frac{n_a}{p_a}\right)\le\sum_{cyc}\left(\frac{n_a}{h_a}+2\right)\overset{CBS}{\le}6+\sqrt{\sum_{cyc}\frac{1}{h_a}\cdot\sum_{cyc}\frac{{n_a}^2}{h_a}}$$

$$=6+\sqrt{\frac{1}{r}\cdot\sum_{cyc}\frac{a}{2sr}\cdot s\left(s-a+\frac{(b-c)^2}{a}\right)}$$

$$=6+\frac{1}{\sqrt{2}r}\cdot\sqrt{\sum_{cyc}[a(s-a)+(b-c)^2]}=6+\frac{1}{\sqrt{2}r}\cdot\sqrt{2r(4R+r)+2(s^2-3r^2-12Rr)}$$

$$=6+\frac{\sqrt{s^2-8Rr-2r^2}}{r}\overset{Gerretsen}{\le}6+\frac{\sqrt{4R^2-4Rr+r^2}}{r}=5+\frac{2R}{r}.$$

as desired. Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuştei, Mohamed Amine Ben Ajiba,

"SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE" – www.ssmrmh.ro