ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$m_a + w_a + h_a \le \left(\frac{1}{\sin\frac{A}{2}} + 1\right)\left(2r + \frac{R}{2}\right)$$

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$$w_a = \frac{2bc}{b+c}.\cos\frac{A}{2} = \frac{4R^2(\cos(B-C)+\cos A).\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$= \frac{R\left(2\cos^2\frac{B-C}{2} - 1 + 1 - 2\sin^2\frac{A}{2}\right)}{\cos\frac{B-C}{2}}$$

$$\Rightarrow w_a = \frac{2R(c^2-s^2)}{c}\left(c = \cos\frac{B-C}{2}\ \textit{and}\ s = \sin\frac{A}{2}\right) \to (m)$$

Case 1 $A < \frac{\pi}{2}$ and then, via (m), $m_a + w_a + h_a \le 2R\cos^2\frac{A}{2} + \frac{2R(c^2-s^2)}{c}$

$$+2R(c^2-s^2) = 2R\left(1 - s^2 + \frac{c^2-s^2}{c} + c^2 - s^2\right) = 2R.\frac{c + c^2 + c^3 - s^2(2c+1)}{c}$$

$$\overset{?}{\le} \left(\frac{s+1}{s}\right)\left(4Rs(c-s) + \frac{R}{2}\right)$$

$$\Leftrightarrow \frac{2\left(c + c^2 + c^3 - s^2(2c+1)\right)}{c} \overset{?}{\le} \frac{(1+s)(1+8sc-8s^2)}{2s}$$

$$\Leftrightarrow -4c^3s + 8c^2s^2 + 4c^2s - 8cs^2 + 4s^3 - 3cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow (4c^2s - 8cs^2 + 4s^3) - (4c^3s - 8c^2s^2 + 4cs) + cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(c^2 - 2cs + s^2) - 4sc\left((c^2 - 2cs + s^2) - s^2 + 1\right) + cs + c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(c-s)^2 - 4sc(c-s)^2 - 4sc(1+s)(1-s) + c(1+s) \overset{?}{\ge} 0$$

$$\Leftrightarrow (4s - 4sc)(c-s)^2 + c(1+s)\left(1 - 4s(1-s)\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow 4s(1-c)(c-s)^2 + c(1+s)(2s-1)^2 \overset{?}{\ge} 0 \to \text{true} \because 1 - c = 1 - \cos\frac{B-C}{2}$$

$$\ge 0, \text{with equality iff } \cos\frac{B-C}{2} = 1\ \textit{and}\ s = \sin\frac{A}{2} = \frac{1}{2} \Rightarrow \text{equality iff}$$

$$B = C\ \textit{and}\ A = \frac{\pi}{3} \Rightarrow \text{equality iff } \Delta\ ABC \text{ is equilateral}$$

Case 2 $A \ge \frac{\pi}{2}$ and then : $4m_a^2 = 2b^2 + 2c^2 - a^2 \overset{?}{\le} a^2 \Leftrightarrow b^2 + c^2 - a^2 \overset{?}{\le} 0$

$$\Leftrightarrow 2bc.\cos A \overset{?}{\le} 0 \to \text{true} \Rightarrow m_a \le \frac{a}{2} = 2R\cos\frac{A}{2}\sin\frac{A}{2} = 2Rs.\sqrt{1-s^2}$$

$$\text{and then, via (m)}, m_a + w_a + h_a \leq$$

$$2Rs.\sqrt{1-s^2} + \frac{2R(c^2-s^2)}{c} + 2R(c^2-s^2) \overset{?}{\leq} \left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)$$

$$= R.\frac{(1+s)(1+8sc-8s^2)}{2s}$$

$$\Leftrightarrow \frac{(1+s)(1+8sc-8s^2)}{2s} - \frac{2(c^2-s^2)}{c} - 2(c^2-s^2) \overset{?}{\geq} 2s.\sqrt{1-s^2}$$

$$\Leftrightarrow \frac{c(1+s)(1+8sc-8s^2) - 4s(c^2-s^2) - 4sc(c^2-s^2)}{2sc} \overset{?}{\geq} 2s.\sqrt{1-s^2}$$

$$\Leftrightarrow \frac{-4c^3s + 8c^2s^2 - 4cs^3 + 4c^2s - 8cs^2 + 4s^3 + c(1+s)}{2sc} \overset{?}{\geq} 2s.\sqrt{1-s^2}$$

$$\Leftrightarrow \frac{-4sc(c^2-2cs+s^2) + 4s(c^2-2cs+s^2) + c(1+s)}{2sc} \overset{?}{\geq} 2s.\sqrt{1-s^2}$$

$$\Leftrightarrow \frac{4s(1-c)(c-s)^2 + c(1+s)}{2sc} \underset{(*)}{\overset{?}{\geq}} 2s.\sqrt{1-s^2}$$

$$\text{Now, } 1 - c = 1 - \cos\frac{B-C}{2} \geq 0 \therefore \text{LHS of } (*) \geq \frac{1+s}{2s} \overset{?}{\geq} 2s.\sqrt{1-s^2}$$

$$\Leftrightarrow (1+s)^2 - 16s^4(1-s^2) \overset{?}{\geq} 0 \Leftrightarrow 16s^6 - 16s^4 + s^2 + 2s + 1 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{1}{16}\left((4s-3)^2\left(16s^4+24s^3+11s^2+3s-\frac{11}{16}\right)+\frac{1}{16}(355-184s)\right) \underset{(**)}{\overset{?}{\geq}} 0$$

$$\text{Now, } A \geq \frac{\pi}{2} \Rightarrow \frac{A}{2} \geq \frac{\pi}{4} \Rightarrow 1 > s \geq \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \because 3s - \frac{11}{16} \text{ is } \uparrow \text{ on } \left[\frac{1}{\sqrt{2}}, 1\right)$$

$$\therefore 3 - \frac{11}{16} > 3s - \frac{11}{16} \geq 3.\frac{1}{\sqrt{2}} - \frac{11}{16} > 0\ \forall s \in \left[\frac{1}{\sqrt{2}}, 1\right) \text{ and } \because s \in \left[\frac{1}{\sqrt{2}}, 1\right)$$

$$\therefore 355 - 184s > 0 \therefore \text{LHS of } (**) > \frac{1}{16}.(4s-3)^2(16s^4+24s^3+11s^2) > 0$$

$$\Rightarrow (**) \text{ is true (strict inequality)} \Rightarrow (*) \text{ is true} \Rightarrow m_a + w_a + h_a <$$

$$\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right) \text{ and combining both cases, } m_a + w_a + h_a \leq$$

$$\left(\frac{1}{\sin\frac{A}{2}}+1\right)\left(2r+\frac{R}{2}\right)\ \forall \text{ ABC, with equality iff } \Delta \text{ ABC is equilateral (QED)}$$