ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationships holds:

$$R \geq \frac{1}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)} \cdot r$$

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}} \cdot r \leq s \leq 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right) \cdot R$$

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$$We\ need\ to\ show:$$

$$R \geq \frac{1}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)} \cdot r$$

$$\frac{4R}{r} \geq \frac{4}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}$$

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \geq \frac{4}{2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right)}\ or\ 1-\sin\frac{A}{2} \geq 2\sin\frac{B}{2}\sin\frac{C}{2}$$

$$1-\sin\frac{A}{2} \geq \cos\frac{B-C}{2} - \cos\frac{B+C}{2}\ or$$

$$1-\sin\frac{A}{2} \overset{A+B+C=\pi}{\geq} \cos\frac{B-C}{2} - \sin\frac{A}{2}\ or\ \cos\frac{B-C}{2} \leq 1\ True$$

$$We\ need\ to\ show:$$

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}} \cdot r \leq s$$

$$\frac{s}{r} \geq \frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\ or, \prod \cot\frac{A}{2} \geq \frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\cot\frac{B}{2}\cot\frac{C}{2}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\ or\ \frac{\left(\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right)}{\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)}\ge\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{\left(\sin\frac{A}{2}+\cos\frac{B-C}{2}\right)}{\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)}\overset{A+B+C=\pi}{\ge}\frac{\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}$$

$$\frac{x+y}{y-x}\overset{\sin\frac{A}{2}=x,\cos\frac{B-C}{2}=y}{\ge}\frac{1+x}{1-x}$$

$$x+y-x^2-xy\ge y-x+xy-x^2\ or\ 2x\ge 2xy\ or\ y\le 1\ or$$

$$\cos\frac{B-C}{2}\le 1\ True\ (A)$$

$$We\ need\ to\ show:$$

$$s\le 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)R\ or,\frac{s}{2R}\le\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$\frac{s}{4R}\le\frac{1}{2}\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$\prod\cos\frac{A}{2}\le\frac{1}{2}\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)$$

$$2\cos\frac{B}{2}\cos\frac{C}{2}\le\left(1+\sin\frac{A}{2}\right)\ or\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\le 1+\sin\frac{A}{2}$$

$$or,\sin\frac{A}{2}+\cos\frac{B-C}{2}\overset{A+B+C=\pi}{\le}1+\sin\frac{A}{2}\ or\cos\frac{B-C}{2}\le 1\ True\ (B)$$

$$From\ (A)\&(B):$$

$$\frac{\cot\frac{A}{2}\left(1+\sin\frac{A}{2}\right)}{1-\sin\frac{A}{2}}\cdot r\le s\le 2\cos\frac{A}{2}\left(1+\sin\frac{A}{2}\right)\cdot R$$