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In any Δ ABC, the following relationship holds :

$$\frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}\geq\frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}=\frac{s^2+2Rr+r^2}{16Rr^2s^2}.\sum_{cyc}w_a\geq\frac{s^2+2Rr+r^2}{16Rr^2s^2}.\sum_{cyc}h_a=$$

$$=\frac{s^2+2Rr+r^2}{16Rr^2s^2}.\frac{s^2+4Rr+r^2}{2R}\overset{?}{\geq}\frac{4}{9}.\frac{(s^2+4Rr+r^2)^2}{16R^2r^2s^2}\Leftrightarrow s^2\overset{?}{\geq}14Rr-r^2\rightarrow\text{true}$$

$$\because s^2\overset{\text{Gerretsen}}{\geq}14Rr-r^2+2r(R-2r)\overset{\text{Euler}}{\geq}14Rr-r^2$$

$$\therefore\frac{1}{w_a w_b}+\frac{1}{w_b w_c}+\frac{1}{w_c w_a}\geq\frac{4}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\ \forall\ ABC,$$

with equality iff Δ ABC is equilateral (QED)