ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$s^2+12Rr+30r^2\le(w_a+w_b+w_c)^2\le s^2+21Rr+12r^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(a\sin\frac{A}{2}\right)=4R\sum_{cyc}\left(\sin^2\frac{A}{2}\cos\frac{A}{2}\right)=R.\sum_{cyc}\left(2\sin^2\frac{A}{2}.\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)$$

$$\overset{0<\cos\frac{B-C}{2}\le 1\text{ and analogs}}{\ge}R.\sum_{cyc}\left((1-\cos A).\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right)\right)$$

$$=R.\sum_{cyc}((1-\cos A).(\sin B+\sin C))$$

$$=R.\sum_{cyc}(\sin B+\sin C)-R\sum_{.cyc}\left(\cos A\left(\sum_{cyc}\sin A-\sin A\right)\right)$$

$$=2R.\frac{s}{R}-R.\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\sin A\right)+\frac{R}{2}.\sum_{cyc}\sin 2A$$

$$=2s-R\left(\frac{R+r}{R}\right)\left(\frac{s}{R}\right)+2R.\frac{4Rrs}{8R^3}=2s-s-\frac{rs}{R}+\frac{rs}{R}\Rightarrow\sum_{cyc}\left(a\sin\frac{A}{2}\right)\ge s\to(m)$$

$$\sum_{cyc}\frac{a}{(b+c)^2}=\sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2}=2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2}-\sum_{cyc}\frac{1}{b+c}$$

$$=\frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2-2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2}-\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}=$$

$$\frac{\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\Rightarrow\sum_{cyc}\frac{a}{(b+c)^2}\overset{(\bullet)}{=}\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Now, }\sum_{cyc}w_a^2=\sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2}=\sum_{cyc}\frac{bc((b+c)^2-a^2)}{(b+c)^2}=\sum_{cyc}\left(bc-\frac{a^2bc}{(b+c)^2}\right)$$

$$\overset{\text{via }(\bullet)}{=}s^2+4Rr+r^2+$$

$$2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}\Rightarrow(w_a+w_b+w_c)^2=$$

$$\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{2.16Rr^2s^2}{s^2+2Rr+r^2}\cdot\sum_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{4rs}{s^2+2Rr+r^2}\cdot\sum_{cyc}\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}.(2(s-a)+a)}{\cos\frac{A}{2}}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}\frac{2r\tan\frac{A}{2}\cos\frac{A}{2}}{\tan\frac{A}{2}}+\sum_{cyc}\left(a\sin\frac{A}{2}\right)\right)$$

$$\overset{\text{via (m)}}{\geq}\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}+s\right)$$

$$\overset{0<\cos\frac{B-C}{2}\leq 1\text{ and analogs}}{\geq}\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\sum_{cyc}(\sin B+\sin C)+s\right)$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(4R+r)^2}{(s^2+2Rr+r^2)^2}$$

$$+\frac{16Rrs}{s^2+2Rr+r^2}\cdot\left(\frac{2s}{R}+s\right)\overset{?}{\geq}s^2+12Rr+30r^2$$

$$\Leftrightarrow\frac{16(R+2r)s^2}{s^2+2Rr+r^2}\overset{?}{\geq}\frac{(16R+29r)s^4+r(20R^2+108Rr+58r^2)s^2+r^2(48R^3+152R^2r+124Rr^2+29r^3)}{(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow 3s^4+(12R^2-28Rr-26r^2)s^2-r(48R^3+152R^2r+124Rr^2+29r^3)\overset{?}{\geq}0\quad(1)$$

$$\text{and}\because P=3(s^2-16Rr+5r^2)^2+4(3R^2+17Rr-14r^2)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq}0\therefore\text{in order to prove (1), it suffices to prove : LHS of (1)}\overset{?}{\geq}P$$

$$\Leftrightarrow 36t^3+27t^2-220t+44\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(36t^2+99t-22)\overset{?}{\geq}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\Rightarrow(1)\text{ is true}\therefore(w_a+w_b+w_c)^2\geq s^2+12Rr+30r^2$$

$$\text{We shall now evaluate : }\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\text{ and it's }=\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$=3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$=4s\sum_{cyc}\frac{1}{b+c}-4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2-\frac{2}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}(b+c)\right)$$

$$= \frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - \frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2} + \frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc} \frac{4s(s-a)}{(b+c)^2} \overset{(\bullet\bullet)}{=} \frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m = 2R^2+10Rr-r^2$ and $n = 2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4 - s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(*)}{\leq} 0$$

$$\text{Now, } w_a+w_b+w_c = \frac{\sqrt{bc}}{b+c}.\sqrt{4s(s-a)} \overset{CBS}{\leq} \sqrt{\sum_{cyc} bc}.\sqrt{\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}}$$

$$\overset{via\,(\bullet\bullet)}{=} \sqrt{(s^2+4Rr+r^2).\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}} \overset{?}{\leq} \sqrt{s^2+21Rr+12r^2}$$

$$\Leftrightarrow (R-5r)s^4 + r(8R^2-2Rr+6r^2)s^2 +$$

$r^2(84R^3+116R^2r+61Rr^2+11r^3) \overset{?}{\geq} 0$ (2) and it's trivially true if : $R-5r\geq 0$

and so, we now focus on the *case* when : $R-5r<0$ *and* then, *via* $(*)$,

$$(R-5r)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)\geq 0$$

$\therefore$ in order to prove (2), it suffices to prove : LHS of (2) $\geq$

$$(R-5r)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right) \Leftrightarrow$$

$$(R^3+2R^2r-26Rr^2+4r^3)s^2 \overset{?}{\geq} r(16R^4-89R^3r-86R^2r^2-30Rr^3-4r^4) \quad (3)$$

Case 1 $R^3+2R^2r-26Rr^2+4r^3\geq 0$ *and* then : LHS of (3) $\overset{Gerretsen}{\geq}$

$$(R^3+2R^2r-26Rr^2+4r^3)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } (3)$$

$$\Leftrightarrow 29t^3-85t^2+56t-4 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(29t^2-27t+2) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow$ (3) is true

Case 2 $R^3+2R^2r-26Rr^2+4r^3<0$ *and* then : LHS of (3) $\overset{Gerretsen}{\geq}$

$$(R^3+2R^2r-26Rr^2+4r^3)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of } (3)$$

$$\Leftrightarrow t^5-t^4-t^3+t^2-8t+4 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(t^4+t^3+t^2+3t-2) \overset{?}{\geq} 0 \to \text{true}$$

$\because t \overset{Euler}{\geq} 2 \Rightarrow$ (3) is true $\therefore$ combining both *cases*, (3) $\Rightarrow$ (2) is true $\forall \Delta$ ABC

$\therefore (w_a+w_b+w_c)^2 \leq s^2+21Rr+12r^2$ *and* so,

$s^2+12Rr+30r^2 \leq (w_a+w_b+w_c)^2 \leq s^2+21Rr+12r^2$ $\forall$ ABC,

with equ*ality* iff Δ ABC is equi*lateral* (QED)