ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$2m_a(n_a + w_a + g_a) \ge 3r_b r_c + n_a^2 + w_a^2 + g_a^2$$

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$$an_a^2 . ag_a^2 \overset{?}{\ge} a^2s^2(s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\begin{pmatrix} b^2(s-b)+c^2(s-c) \\ -a(s-b)(s-c) \end{pmatrix} \underset{(*)}{\overset{?}{\ge}} a^2s^2(s-a)^2$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$ and via such substitutions, $(*) \Leftrightarrow$

$$\left(z(z+x)^2+y(x+y)^2-yz(y+z)\right)\left(y(z+x)^2+z(x+y)^2-yz(y+z)\right)$$

$$\ge x^2(y+z)^2(x+y+z)^2 \Leftrightarrow xy^2+xz^2+y^3+z^3 \ge 2xyz+yz(y+z)$$

$$\Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \ge 0 \to \text{true} \Rightarrow (*) \text{ is true} \Rightarrow n_a g_a \ge s(s-a) \to (1)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$

and $b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c)$ and $(m)+(n) \Rightarrow$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2) = 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) + a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2 = 2(n_a^2+g_a^2)$$

$$\Rightarrow 2(b-c)^2+4s(s-a) = 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(\bullet)}{=} (b-c)^2+2s(s-a)$$

$$\Rightarrow n_a^2+g_a^2+2n_ag_a \overset{\text{via } (2)}{\ge} (b-c)^2+4s(s-a) \Rightarrow (n_a+g_a)^2 \ge 4m_a^2$$

$$\Rightarrow n_a+g_a \ge 2m_a \to (2)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$

$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$ $= an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2 \Rightarrow an_a^2 = as^2 +$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s-\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(\bullet\bullet)}{=} s(s-a)+\frac{s(b-c)^2}{a}$$

Via $(\bullet)$ and $(\bullet\bullet)$, $g_a^2 = (b-c)^2+2s(s-a)-s^2+\dfrac{4s(s-b)(s-c)}{a}$

$$= s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a)^2 + \frac{(s-a)(a^2 - (b-c)^2)}{a} \Rightarrow g_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$\therefore$ via ②, $(\bullet\bullet)$ and $(\bullet\bullet\bullet)$, we have : $2m_a(n_a + w_a + g_a) - (3r_b r_c + n_a^2 + w_a^2 + g_a^2)$

$$\geq 2m_a w_a + 4m_a^2 - 3s(s-a) - s(s-a) - \frac{s(b-c)^2}{a} - s(s-a)$$

$$+ \frac{s(s-a)(b-c)^2}{(2s-a)^2} - s(s-a) + \frac{(s-a)(b-c)^2}{a}$$

$$\overset{\text{Lascu} + \text{A-G}}{\geq} 2s(s-a) + 4s(s-a) + (b-c)^2 - 3s(s-a) - s(s-a) - \frac{s(b-c)^2}{a}$$

$$-s(s-a) + \frac{s(s-a)(b-c)^2}{(2s-a)^2} - s(s-a) + \frac{s(b-c)^2}{a} - (b-c)^2$$

$$= \frac{s(s-a)(b-c)^2}{(2s-a)^2} \geq 0 \therefore 2m_a(n_a + w_a + g_a) \geq 3r_b r_c + n_a^2 + w_a^2 + g_a^2$$

$\forall\, \Delta\, ABC$, " = " iff $b = c$ (QED)