

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC , the following relationship holds :

$$4h_b h_c - r_b r_c \leq 9rr_a$$

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$$\begin{aligned}4h_b h_c - r_b r_c \leq 9rr_a &\Leftrightarrow \frac{16r^2 s^2}{bc} - s(s-a) \leq \frac{9r^2 s}{s-a} \\&\Leftrightarrow 16s(s-a)(s-b)(s-c) - bcs(s-a) \leq 9bc(s-b)(s-c) \\&\Leftrightarrow 16s(s-a)(-s(s-a) + bc) - bcs(s-a) \leq 9bc(-s(s-a) + bc) \\&\Leftrightarrow 16s^2(s-a)^2 - 24bcs(s-a) + 9b^2c^2 \geq 0 \Leftrightarrow (4s(s-a) - 3bc)^2 \geq 0 \\&\Leftrightarrow ((b+c)^2 - a^2 - 3bc)^2 \geq 0 \Leftrightarrow (b^2 + c^2 - a^2 - bc)^2 \geq 0 \\&\Leftrightarrow (2bc \cos A - bc)^2 \geq 0 \Leftrightarrow 4b^2c^2 \left(\cos A - \frac{1}{2}\right)^2 \geq 0 \\&\quad \rightarrow \text{true} \therefore 4h_b h_c - r_b r_c \leq 9rr_a \\&\quad \forall \Delta ABC, " = " \text{ iff } \hat{A} = 60^\circ \text{ (QED)}\end{aligned}$$